

# Development of intelligent systems (RInS)

## Transformations between coordinate frames

Danijel Skočaj

University of Ljubljana

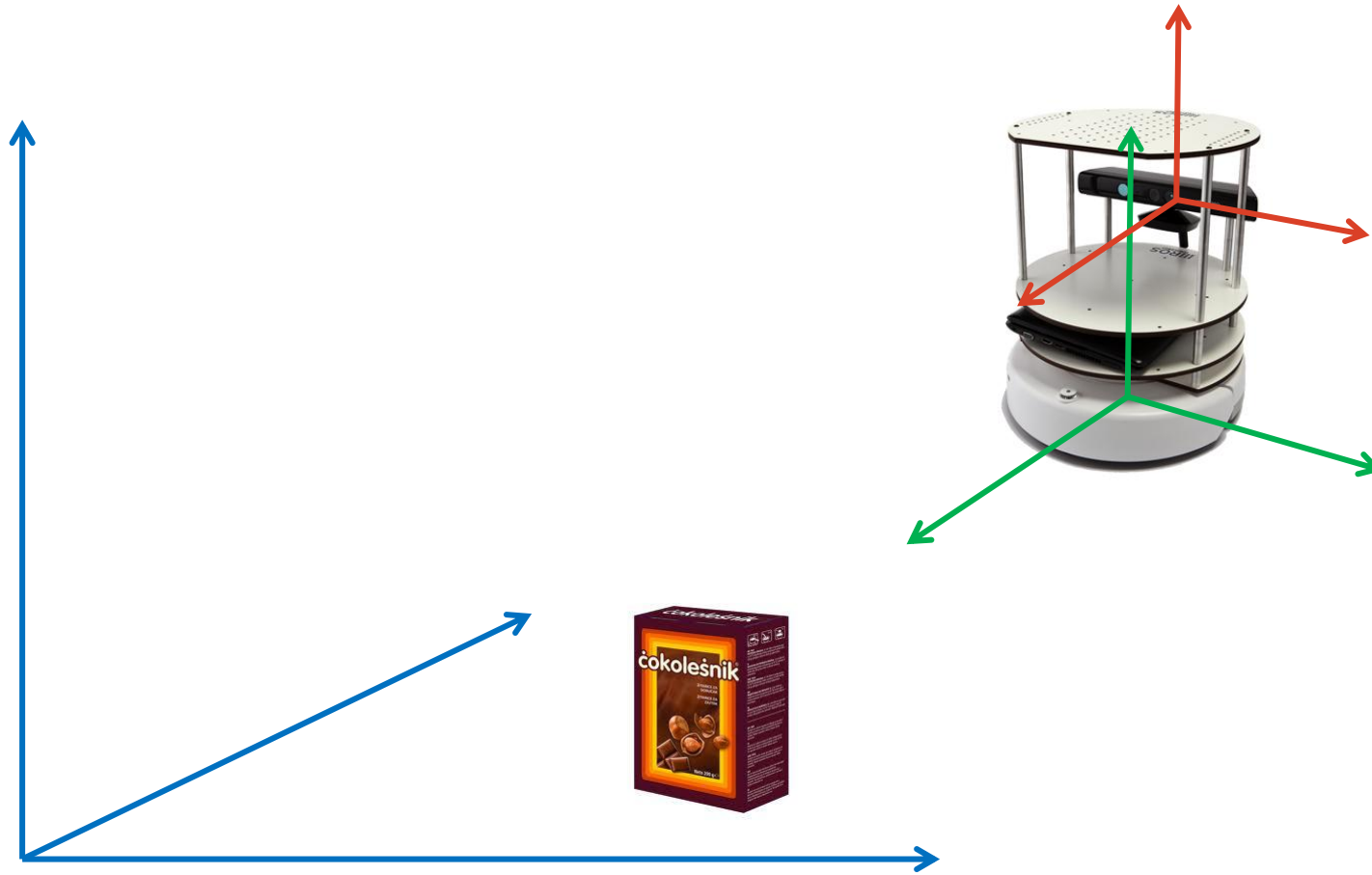
Faculty of Computer and Information Science

Literature: Tadej Bajd (2006).

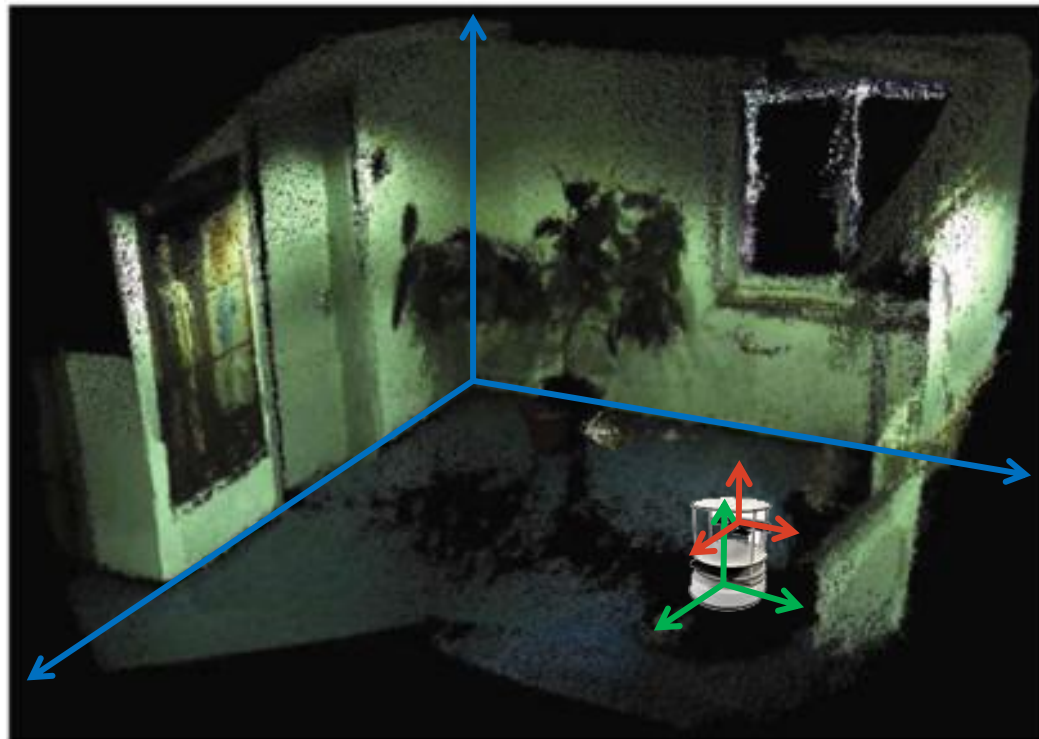
Osnove robotike, chapter 2

Academic year: 2025/26

# Coordinate frames



# 3D environment

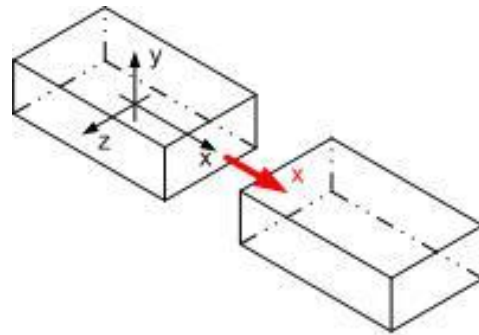


# 2D navigation

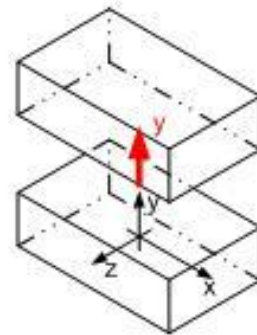


# Degrees of freedom

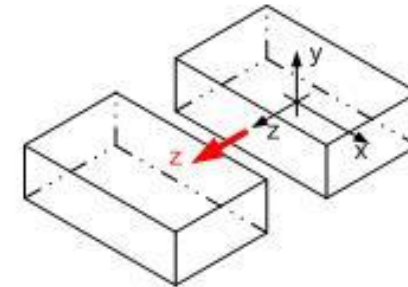
- DOF
- 6 DOF for full description of the pose of an object in space
  - 3 translations (position)
  - 3 rotations (orientation)



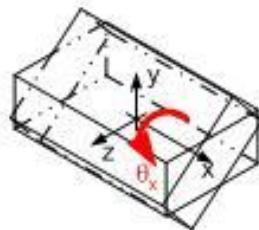
Linear in x-direction



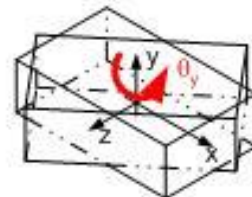
Linear in y-direction



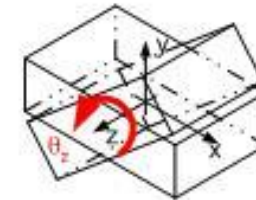
Linear in z-direction



Rotation around x-axis

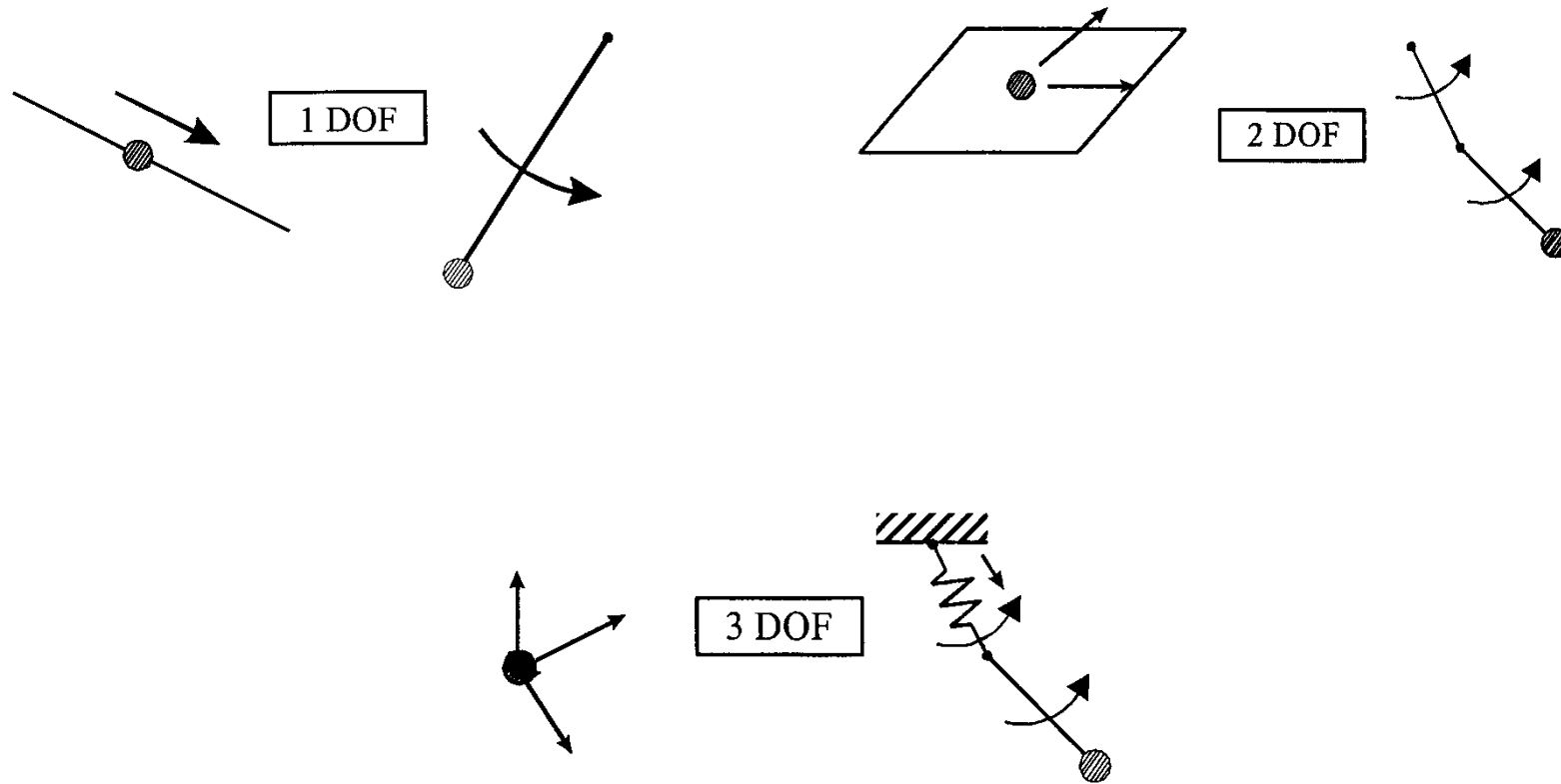


Rotation around y-axis

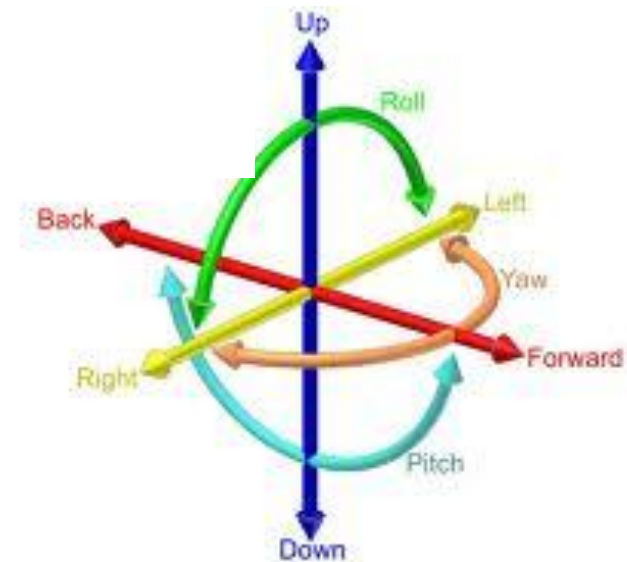
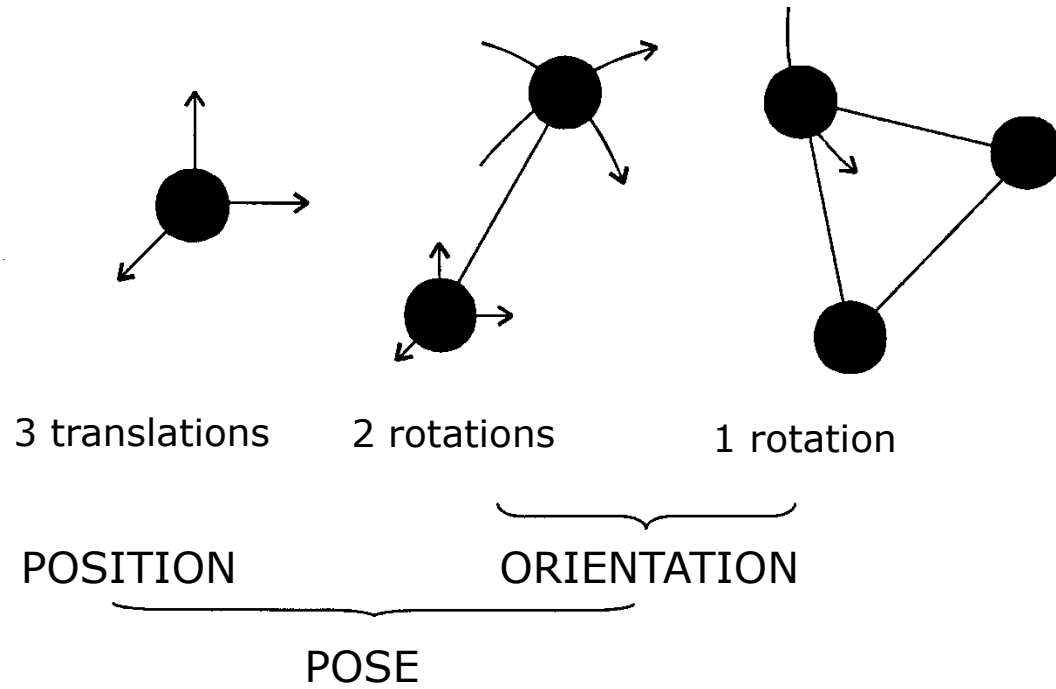


Rotation around z-axis

# Degrees of freedom



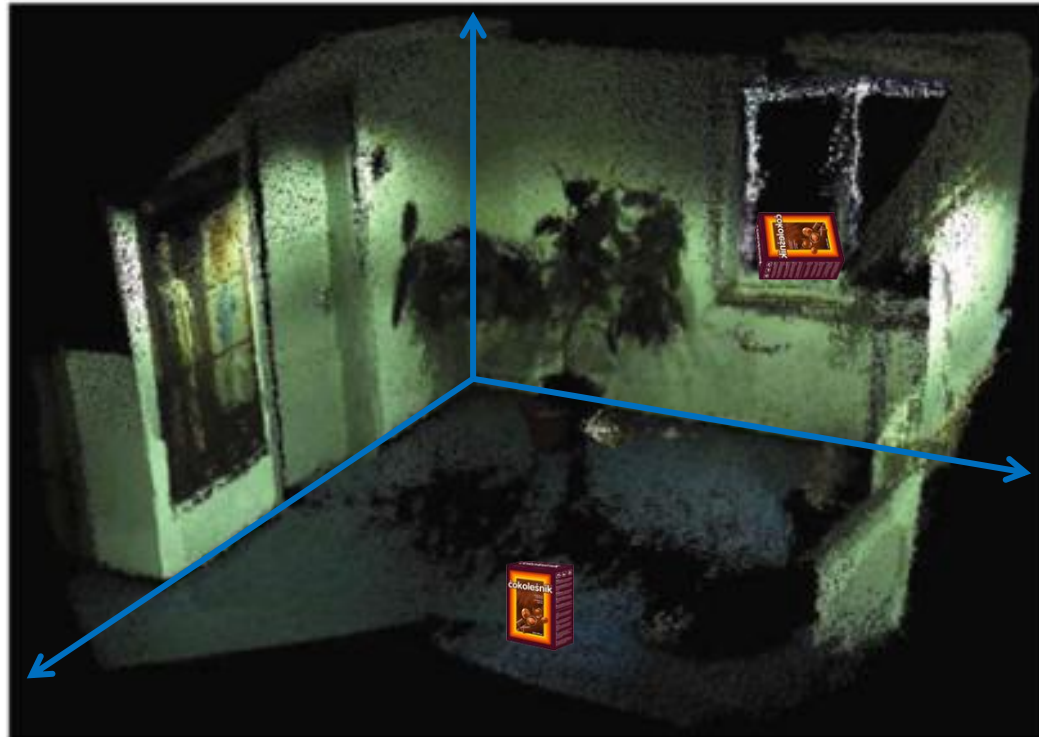
# Degrees of freedom



# Position and orientation of the robot

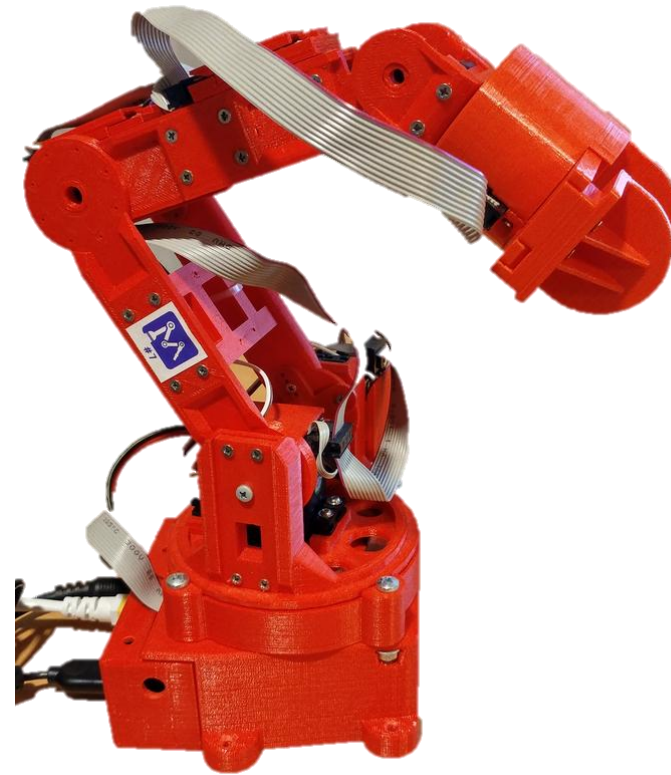


# Pose of the object in 3D space



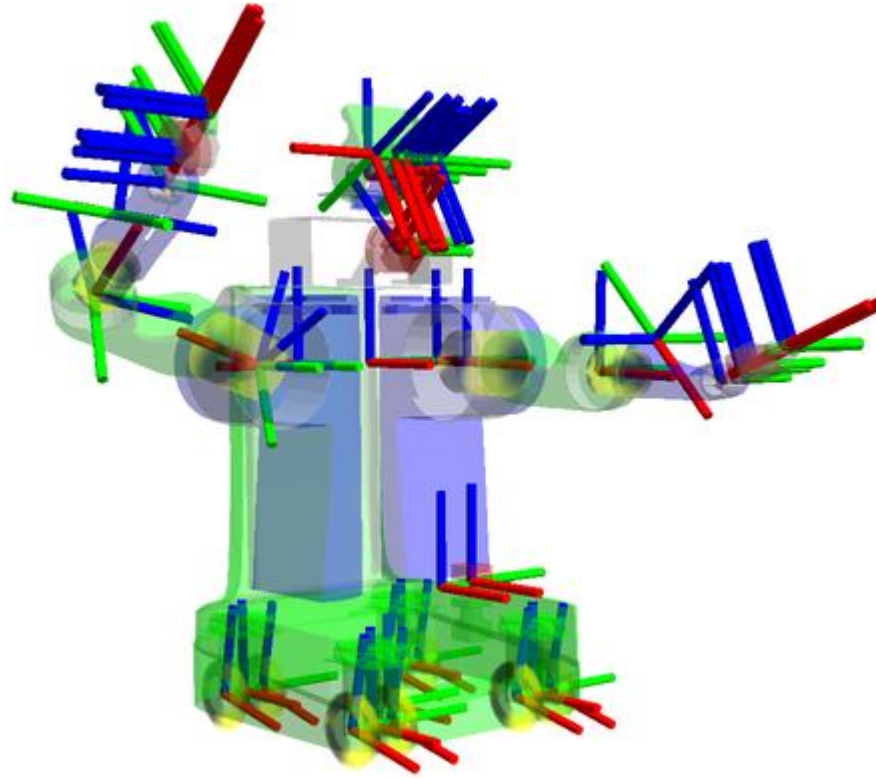
# Robot manipulator

- ViCoS LCLWOS robot manipulator
  - 5DOF
- 6DOF needed for general grasping



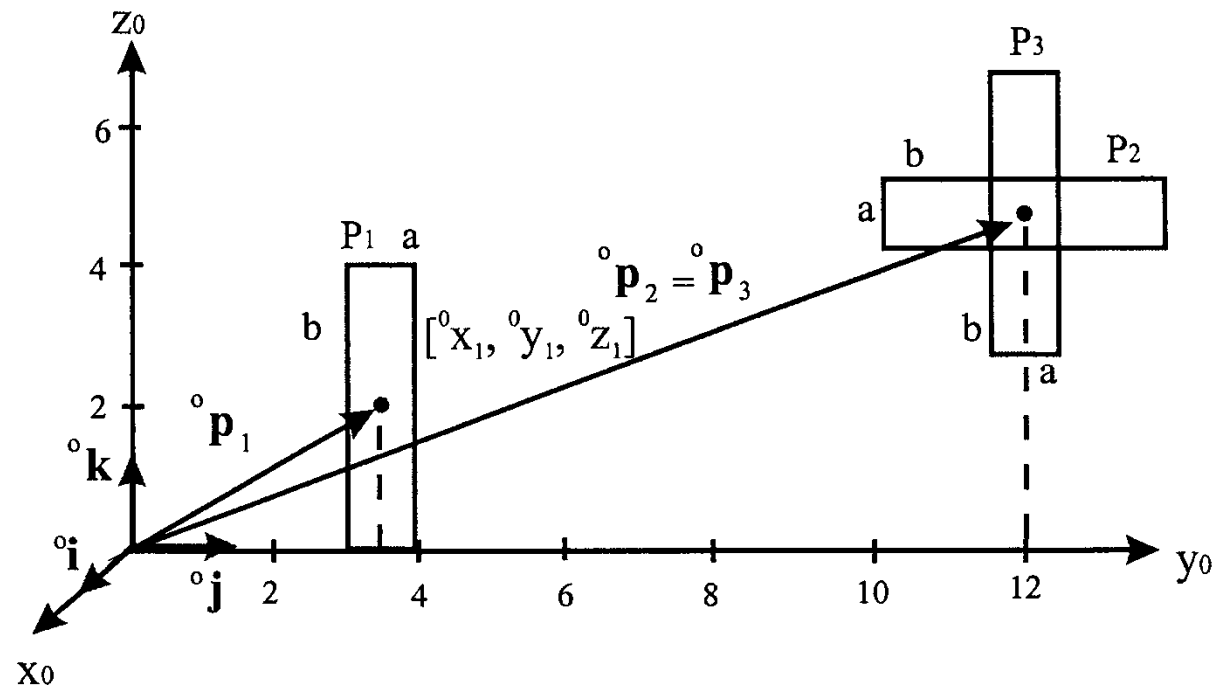
# Chains of coordinate frames

- Transformations between coordinate frames



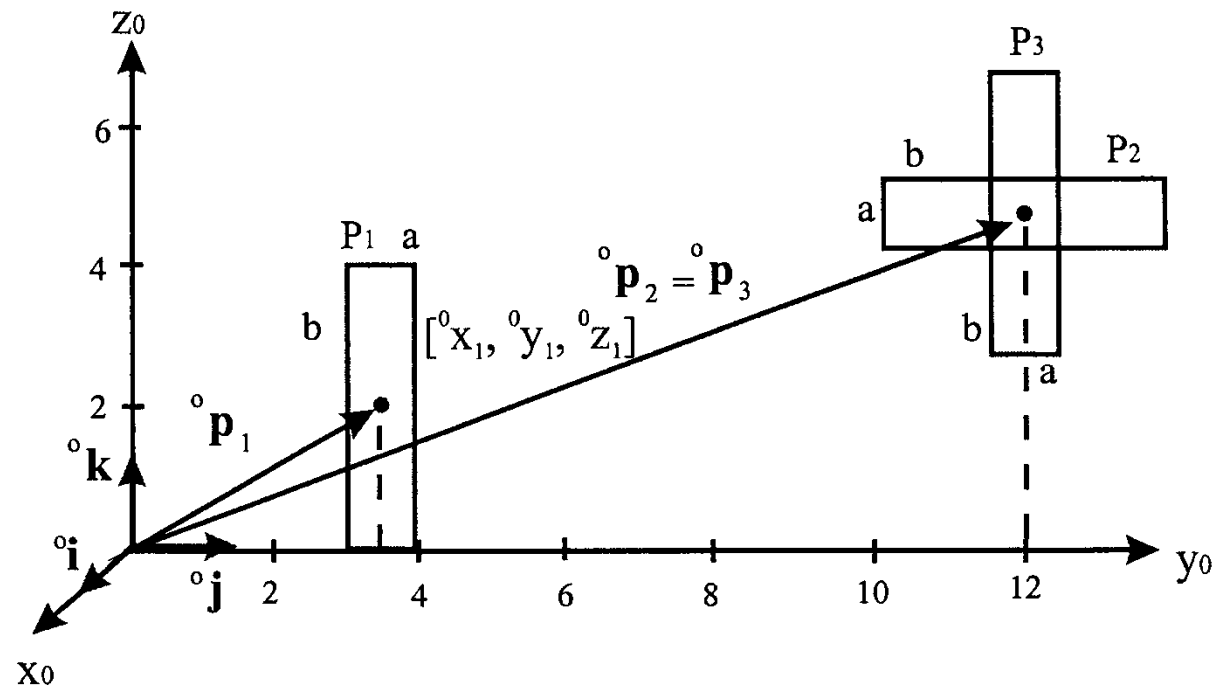
# Position and orientation

- Pose=Position+Orientation
  - Position(P2)=Position (P3)
  - Position(P1)~=Position (P2)
  - Orientation(P1)=Orientation (P3)
  - Orientation(P2)~=Orientation (P3)
  - Pose(P1)~=Pose(P2)~=Pose(P3)



# Translation and rotation

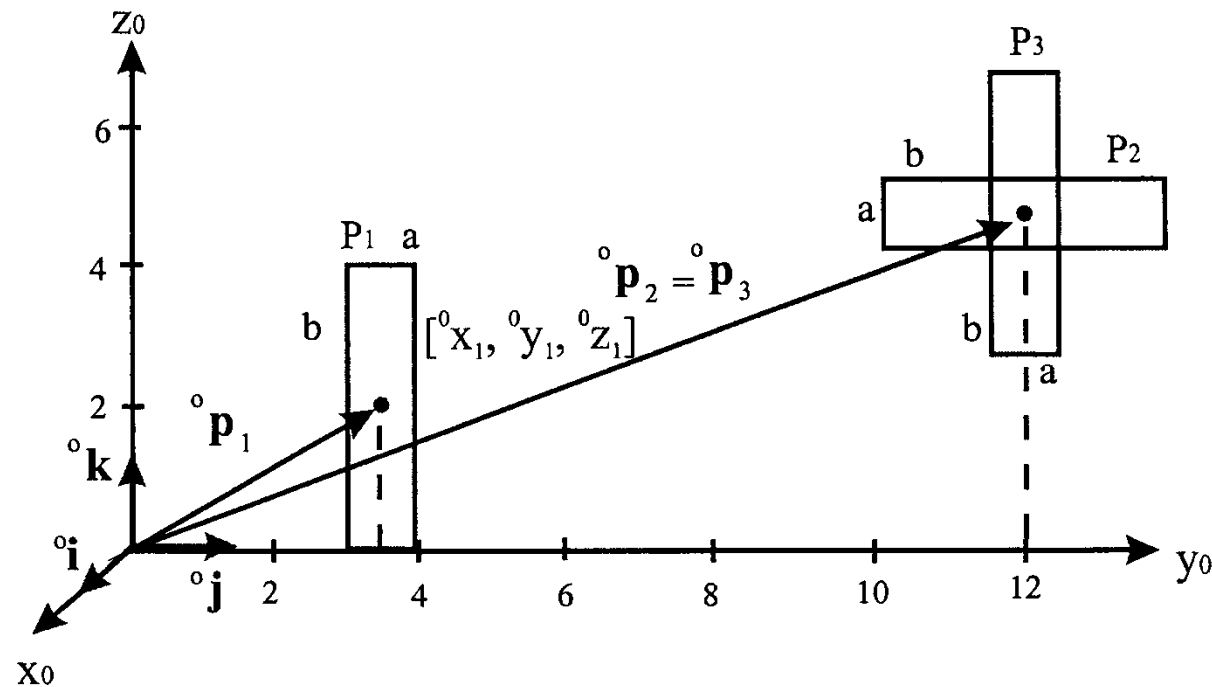
- Moving objects:
  - P1 v P3: Translation (T)
  - P2 v P3: Rotation (R)
  - P1 v P2: Translation in rotation



# Position

- Position: vector from the origin of the coordinate frame to the point
- Position of the object P1:

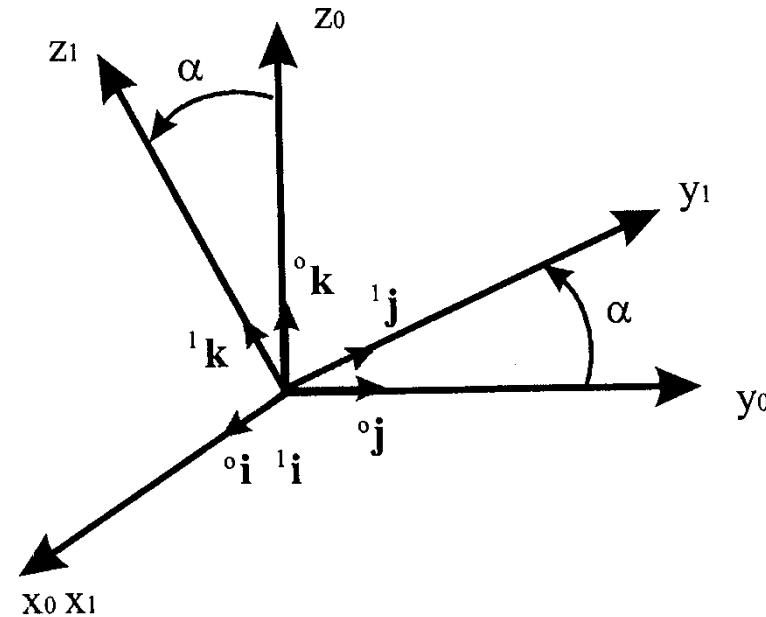
$${}^0\mathbf{p}_1 = {}^0x_1 {}^0\mathbf{i} + {}^0y_1 {}^0\mathbf{j} + {}^0z_1 {}^0\mathbf{k}$$



# Orientation

- Right-handed coordinate frame
- Rotation around  $x_0$  axis:
- Rotation matrix:

$${}^0\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$



- Orientation of c.f.  $O_1$  with respect to c.f.  $O_0$
- Transformation of the vector  ${}^1\mathbf{p}$  expressed in the c.f.  $O_1$  into the coordinates expressed in the c.f.  $O_0$  :

$${}^0\mathbf{p} = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{p}$$

# Rotation matrices

- Rotation around x axis:

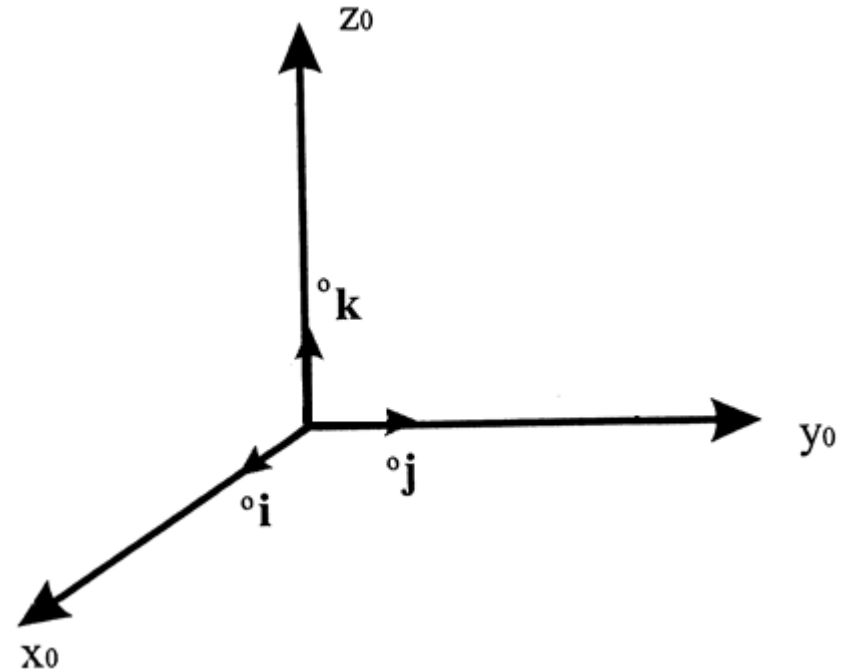
$$\mathbf{R}_{X,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

- Rotation around y axis :

$$\mathbf{R}_{Y,\alpha} = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}$$

- Rotation around z axis :

$$\mathbf{R}_{Z,\alpha} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Properties of rotation matrix

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- Rotation is an orthogonal transformation matrix
- Inverse transformation:

$${}^1\mathbf{R}_0 = ({}^0\mathbf{R}_1)^{-1} = ({}^0\mathbf{R}_1)^T$$

- In the right-handed coordinate frame the determinant equals to 1
- Addition of angles:

$$\mathbf{R}_{X,\alpha_1} \cdot \mathbf{R}_{X,\alpha_2} = \mathbf{R}_{X,\alpha_1+\alpha_2}$$

- Backward rotation:

$$\mathbf{R}_{X,\alpha}^{-1} = \mathbf{R}_{X,-\alpha}$$

# Consecutive rotations

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- Postmultiply the vector with the rotation matrix

- Consecutive rotations:

$${}^0\mathbf{p} = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{p} \qquad {}^1\mathbf{p} = {}^1\mathbf{R}_2 \cdot {}^2\mathbf{p}$$

$${}^0\mathbf{p} = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{R}_2 \cdot {}^2\mathbf{p}$$

- Rotation matrices are postmultiplied:

$${}^0\mathbf{R}_2 = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{R}_2$$

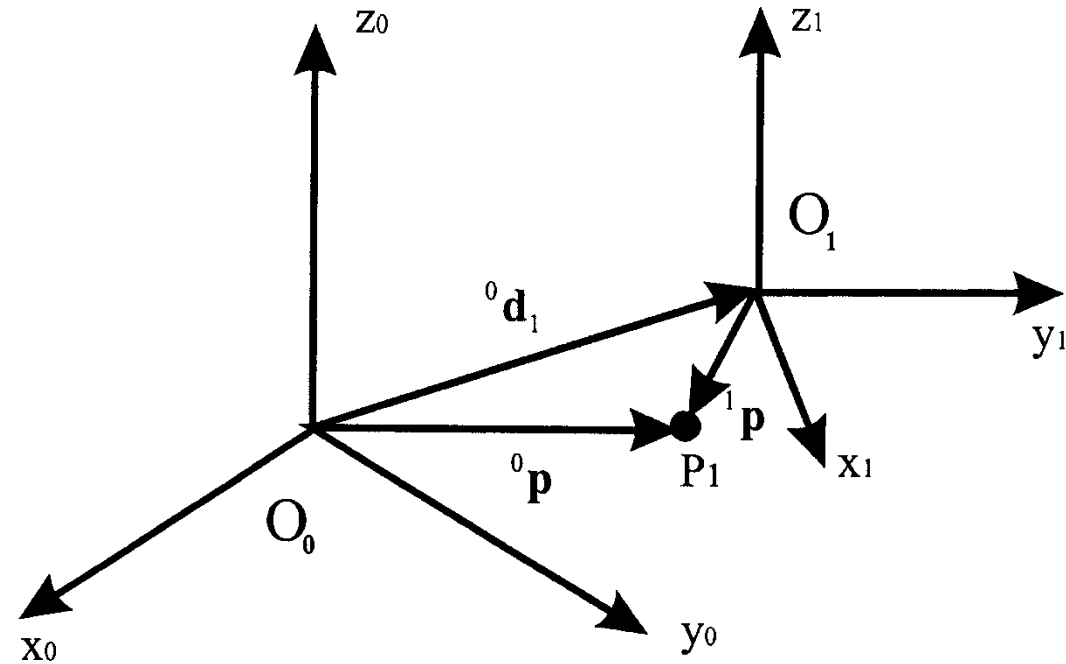
- In general:

- Postmultiply matrices for all rotations
- Rotations always refer to the respective relative current coordinate frame

$${}^0\mathbf{R}_n = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{R}_2 \cdots {}^{n-1}\mathbf{R}_n$$

# Transformations

- Transformation from one c.f. to another:



- If c.f. are parallel:  ${}^0\mathbf{p} = {}^1\mathbf{p} + {}^0\mathbf{d}_1$ 
  - Only translation
- If c.f. are not parallel:  ${}^0\mathbf{p} = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{p} + {}^0\mathbf{d}_1$ 
  - Rotation and translation
  - General pose description

# Matrix notation

- Three coordinate frames:

$${}^0\mathbf{p} = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{p} + {}^0\mathbf{d}_1 \quad {}^1\mathbf{p} = {}^1\mathbf{R}_2 \cdot {}^2\mathbf{p} + {}^1\mathbf{d}_2$$

$${}^0\mathbf{p} = {}^0\mathbf{R}_2 \cdot {}^2\mathbf{p} + {}^0\mathbf{d}_2$$

- Combine the transformations:

$${}^0\mathbf{R}_2 = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{R}_2 \quad {}^0\mathbf{d}_2 = {}^0\mathbf{d}_1 + {}^0\mathbf{R}_1 \cdot {}^1\mathbf{d}_2$$

$${}^0\mathbf{p} = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{R}_2 \cdot {}^2\mathbf{p} + {}^0\mathbf{R}_1 \cdot {}^1\mathbf{d}_2 + {}^0\mathbf{d}_1$$

- We can add the translation vectors if they are expressed in the same coordinate frame
- The two equations in the matrix form:

$$\begin{bmatrix} {}^0\mathbf{R}_1 & {}^0\mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} {}^1\mathbf{R}_2 & {}^1\mathbf{d}_2 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{R}_1 \cdot {}^1\mathbf{R}_2 & {}^0\mathbf{R}_1 \cdot {}^1\mathbf{d}_2 + {}^0\mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix}$$

# Homogeneous transformations

- General pose

$${}^0\mathbf{p} = \mathbf{R} \cdot {}^1\mathbf{p} + \mathbf{d}$$

can be expressed in the matrix form:

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Homogeneous transformation - homogenises (combines) rotation and translation in one matrix
- Very concise and convenient format
- Homogeneous matrix of size 4x4 (for 3D space)
  - One row is added, also 1 in the position vector

$$\begin{bmatrix} {}^0\mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^1\mathbf{p} \\ 1 \end{bmatrix} = {}^0\mathbf{H}_1 \begin{bmatrix} {}^1\mathbf{p} \\ 1 \end{bmatrix}$$

# Homogenous matrix

- Rotation R and translation d:

$$\begin{bmatrix} \begin{matrix} \circ & \circ & \circ \\ \circ & \mathbf{R} & \circ \\ \circ & \circ & \circ \end{matrix} & \begin{matrix} \circ \\ \mathbf{d} \\ \circ \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}$$

- Only rotation:

$$\begin{bmatrix} \begin{matrix} \circ & \circ & \circ \\ \circ & \mathbf{R} & \circ \\ \circ & \circ & \circ \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}$$

- Only translation:

$$\begin{bmatrix} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & \begin{matrix} \circ \\ \mathbf{d} \\ \circ \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}$$

# Properties of homogeneous transformation

- Inverse of homogeneous transformation:

$${}^0\mathbf{p} = \mathbf{R} \cdot {}^1\mathbf{p} + \mathbf{d}$$

$${}^1\mathbf{p} = \mathbf{R}^T \cdot {}^0\mathbf{p} - \mathbf{R}^T \mathbf{d}$$

$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T \cdot \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Consecutive poses:

- Postmultiplication of homogeneous transformations:

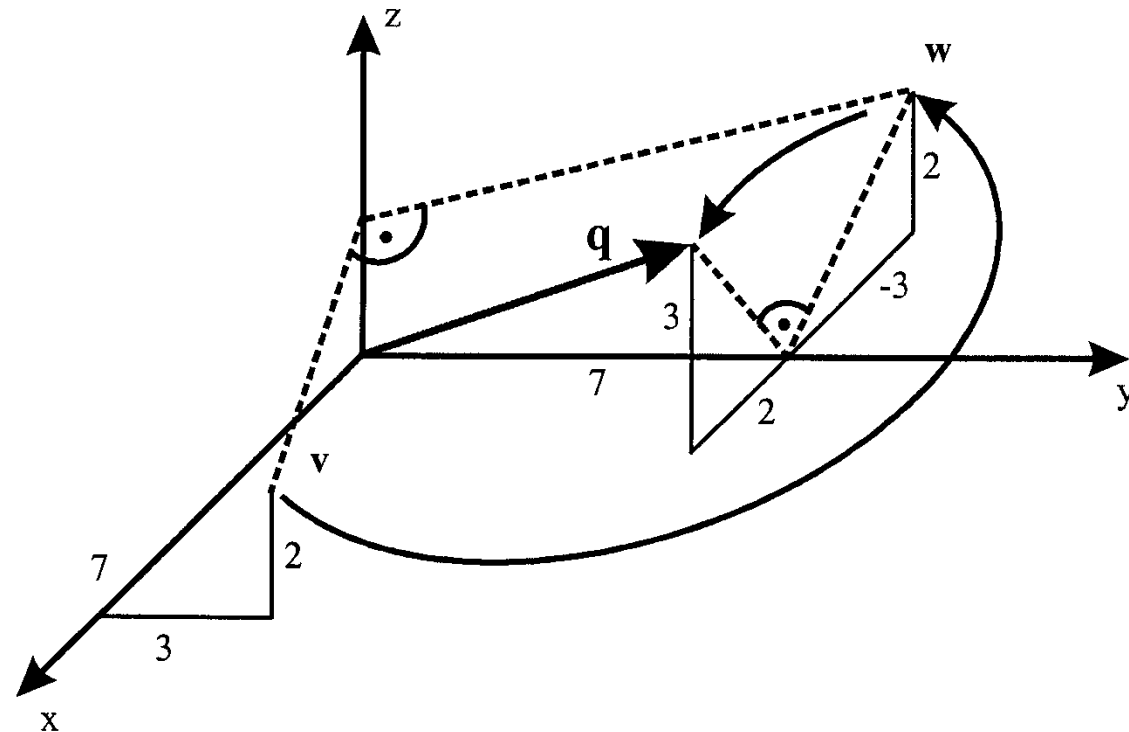
$${}^0\mathbf{H}_2 = {}^0\mathbf{H}_1 \cdot {}^1\mathbf{H}_2$$

$${}^0\mathbf{H}_n = {}^0\mathbf{H}_1 \cdot {}^1\mathbf{H}_2 \dots {}^{n-1}\mathbf{H}_n$$

- An element can be transformed arbitrary number of times – by multiplying homogeneous matrices

# Example

- Two rotations
  - Vector  $\mathbf{v} = [7, 3, 2, 1]^T$   
first rotate for  $90^\circ$  around z axis  $\mathbf{w} = \mathbf{Rot}(z, 90) \mathbf{v}$   
and then for  $90^\circ$  around y axis  $\mathbf{q} = \mathbf{Rot}(y, 90) \mathbf{w}$



# Example– two rotations

$$\mathbf{w} = \mathbf{Rot}(z, 90) \mathbf{v}$$

$$\mathbf{q} = \mathbf{Rot}(y, 90) \mathbf{w}$$

$$\mathbf{q} = \mathbf{Rot}(y, 90) \mathbf{Rot}(z, 90) \mathbf{v}$$

$$\mathbf{Rot}(y, 90) \mathbf{Rot}(z, 90) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 1 \end{bmatrix}$$

# Example - translation

- After two rotations also translate the vector for (4,-3,7)
  - Merge
    - Translation **Trans(4i -3j + 7k)**  
with rotations **Rot(y,90) · Rot(z, 90)**

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$
$$= \mathbf{Trans} (4, -3, 7) \mathbf{Rot} (y,90) \mathbf{Rot} (z, 90)$$

- Transformation of the point (7,3,2):

$$\mathbf{x} = \mathbf{H}_1 \cdot \mathbf{v} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

# Transformation of the coordinate frame

- Homogeneous transformation matrix transforms the base coordinate frame

$$\mathbf{Trans}(4, -3, 7) \mathbf{Rot}(y, 90) \mathbf{Rot}(z, 90)$$

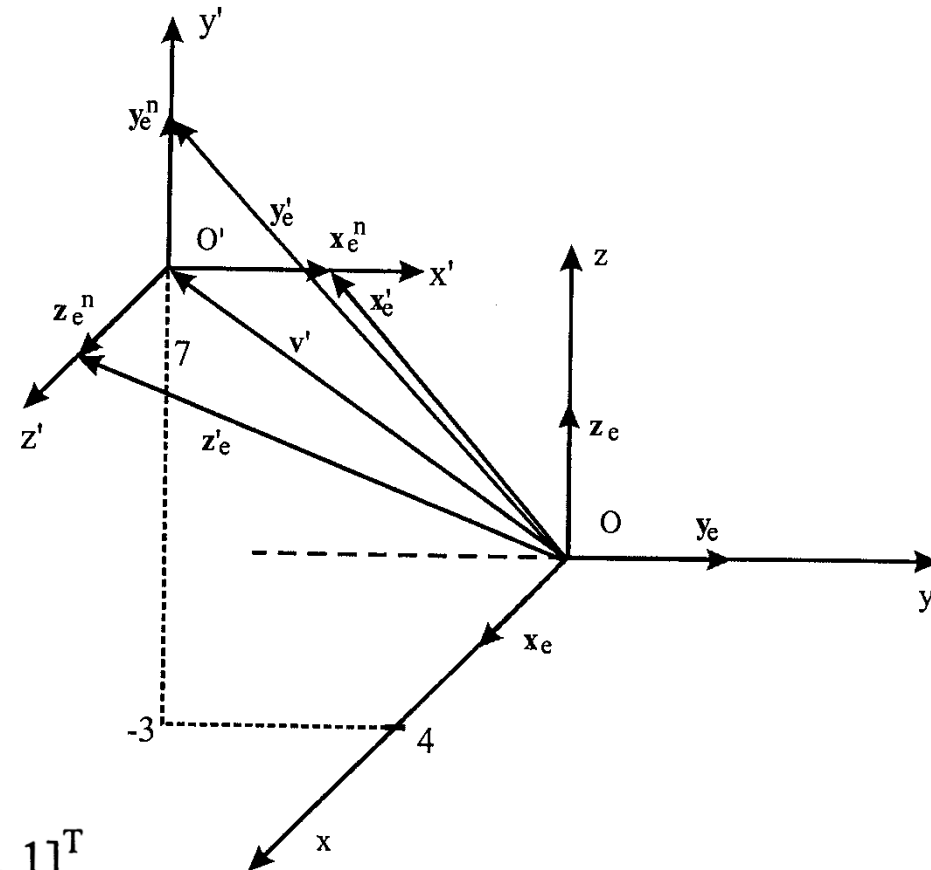
- Vector of origin of c.f.:

$$\mathbf{H}_1 \cdot \mathbf{v} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 7 \\ 1 \end{bmatrix} = \mathbf{v}'$$

- Unit vectors:

$$\begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 7 \\ 1 \end{bmatrix} = \mathbf{x}'_e$$

$$\mathbf{y}'_e = [4, -3, 8, 1]^T, \quad \mathbf{z}'_e = [5, -3, 7, 1]^T$$



# Pose of the coordinate frame

- Unit vectors of the new coordinate frame:

$$\mathbf{x}_e^n : 4\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} - 4\mathbf{i} + 3\mathbf{j} - 7\mathbf{k} = 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{x}_e^n = [0, 1, 0, 0]^T$$

$$\mathbf{y}_e^n : 4\mathbf{i} - 3\mathbf{j} + 8\mathbf{k} - 4\mathbf{i} + 3\mathbf{j} - 7\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k}$$

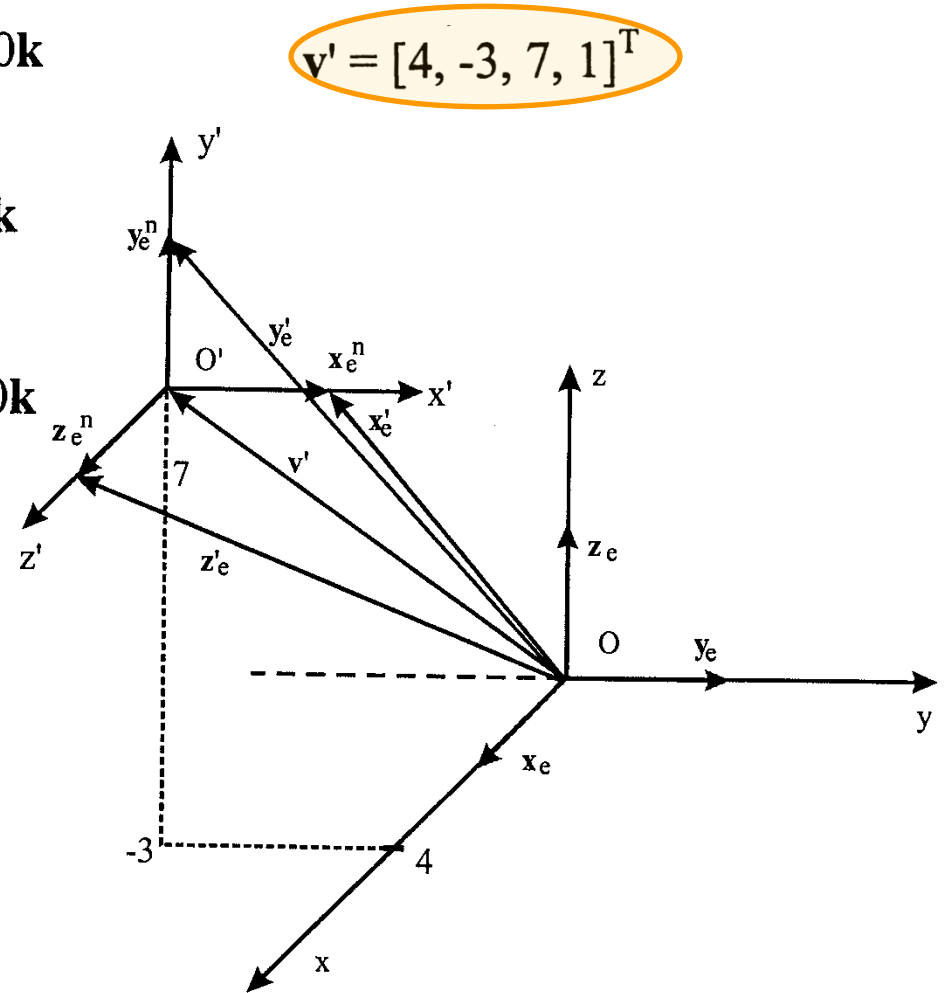
$$\mathbf{y}_e^n = [0, 0, 1, 0]^T$$

$$\mathbf{z}_e^n = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} - 4\mathbf{i} + 3\mathbf{j} - 7\mathbf{k} = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{z}_e^n = [1, 0, 0, 0]^T$$

- Transformation matrix describes the coordinate frame!

$$\begin{bmatrix} \mathbf{x}_e^n & \mathbf{y}_e^n & \mathbf{z}_e^n & \\ 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Movement of the coordinate frame

- Premultiplication or postmultiplication (of an object or c.f.) with transformation

- Example:

- Coordinate frame:

$$\mathbf{C} = \begin{matrix} & \mathbf{i}_c & \mathbf{j}_c & \mathbf{k}_c & \\ \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \mathbf{i} \\ & \mathbf{j} \\ & \mathbf{k} \end{matrix}$$

- Transformation:

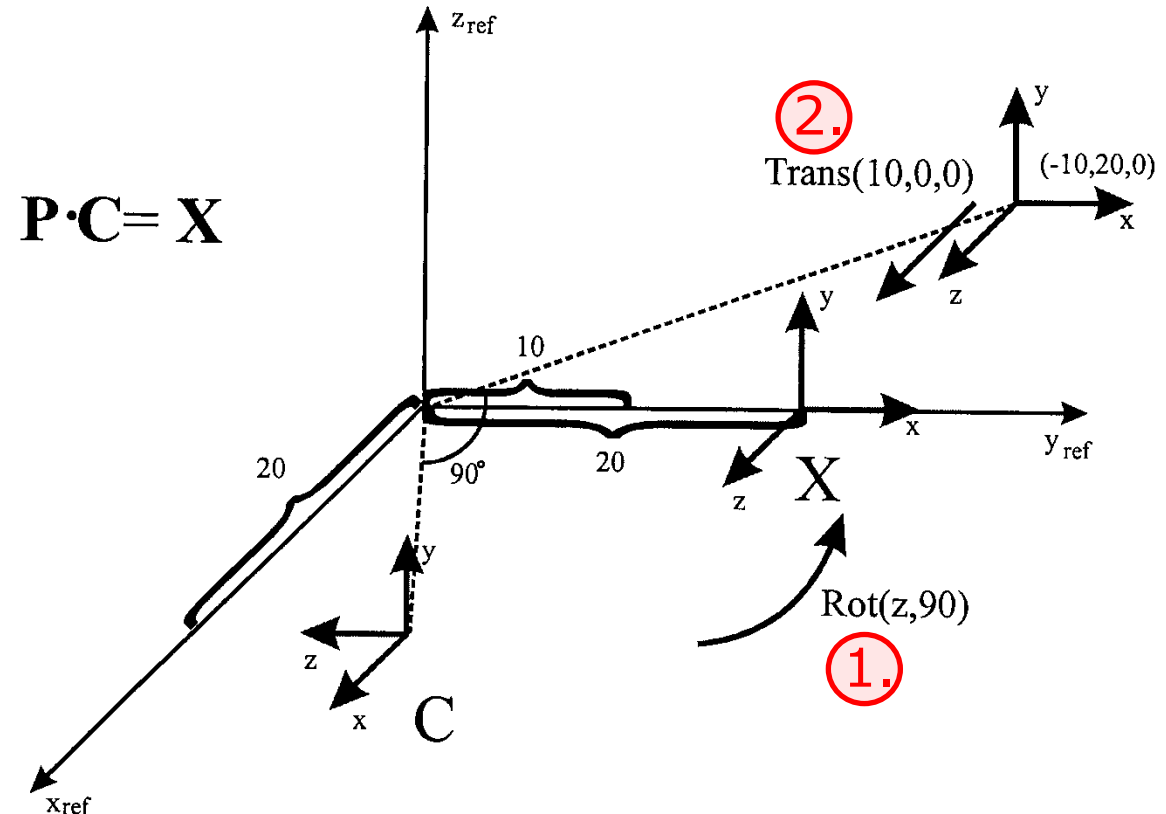
$$\mathbf{P} = \begin{bmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{Trans}(10,0,0) \cdot \mathbf{Rot}(z,90)$$

# Premultiplication

$$\textcircled{P} \cdot \mathbf{C} = \mathbf{X} = \begin{bmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The pose of the object is transformed with respect to the **fixed reference** coordinate frame in which the object coordinates were given.
- Order of transformations:

$$\text{Trans}(10,0,0) \cdot \text{Rot}(z,90)$$

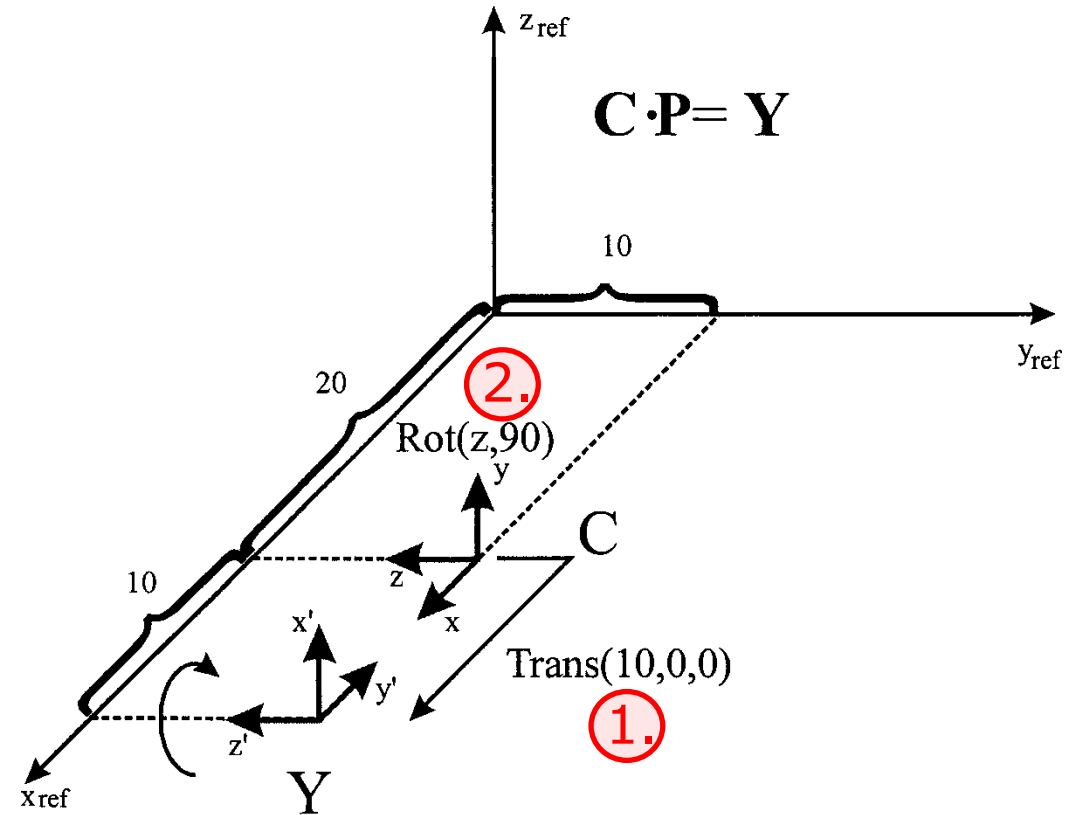


# Postmultiplication

$$\underline{\mathbf{C}} \cdot \mathbf{P} = \mathbf{Y} = \begin{matrix} & \mathbf{i}_c & \mathbf{j}_c & \mathbf{k}_c \\ \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \mathbf{i} & \mathbf{j} & \mathbf{k} \end{matrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 30 \\ 0 & 0 & -1 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The pose of the object is transformed with respect to its **own relative current** coordinate frame
- Order of transformations:

$$\underline{\text{Trans}(10,0,0)} \cdot \underline{\text{Rot}(z,90)}$$



# Movement of the reference c.f.

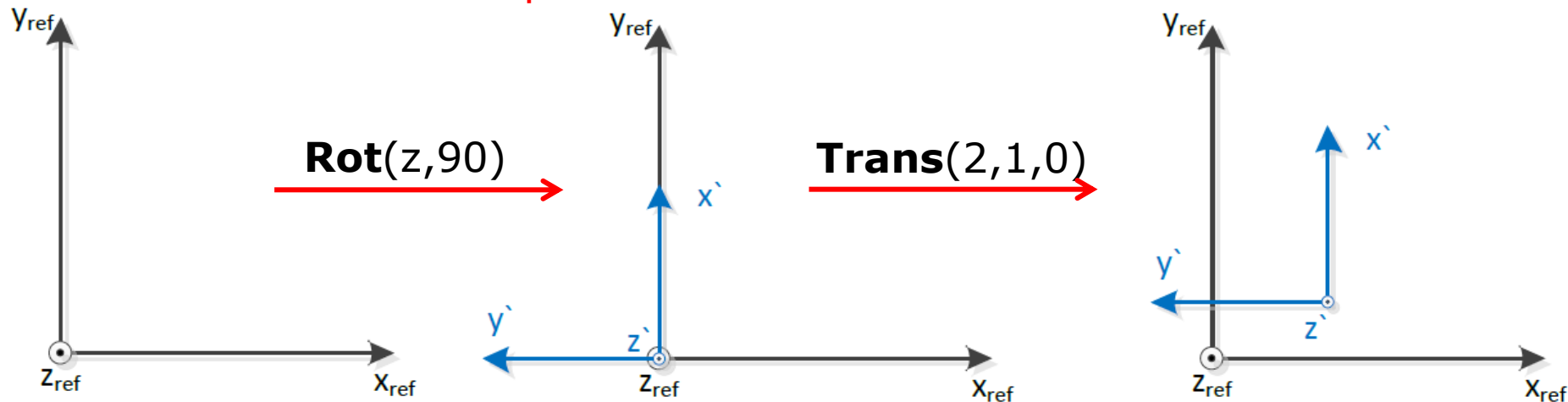
- Example: **Trans(2,1,0)Rot(z,90)**

$$\begin{aligned} \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} &= \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

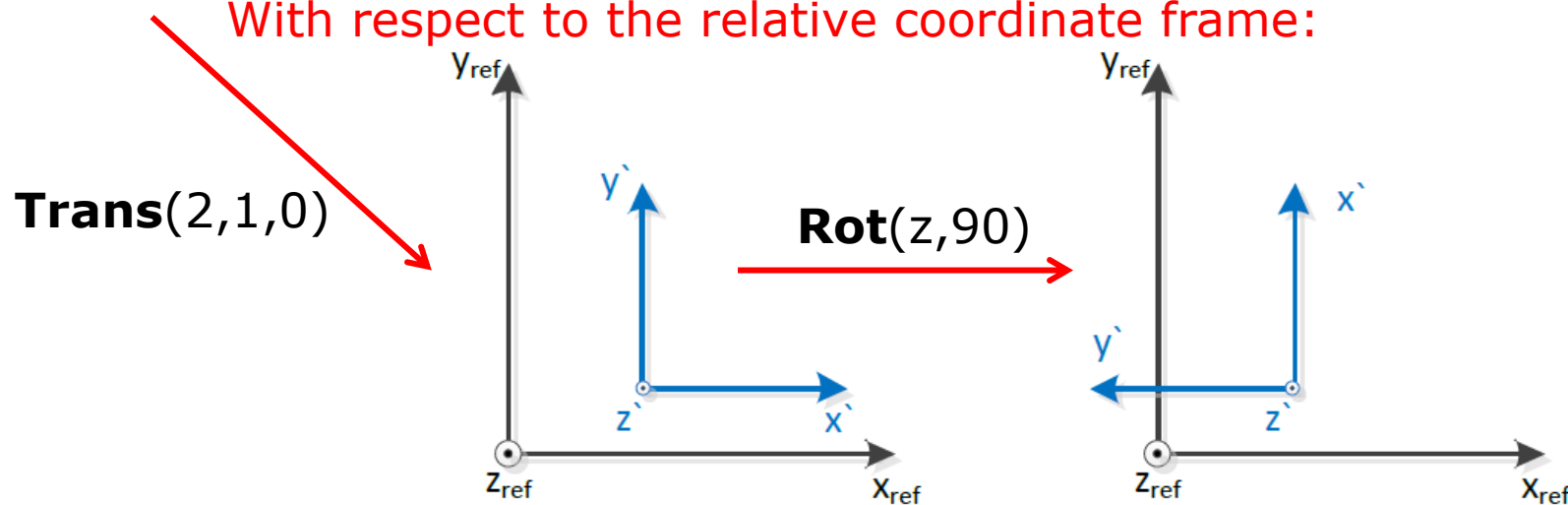
# Movement of the reference c.f.

- Example: **Trans(2,1,0)Rot(z,90)**

With respect to the reference coordinate frame:

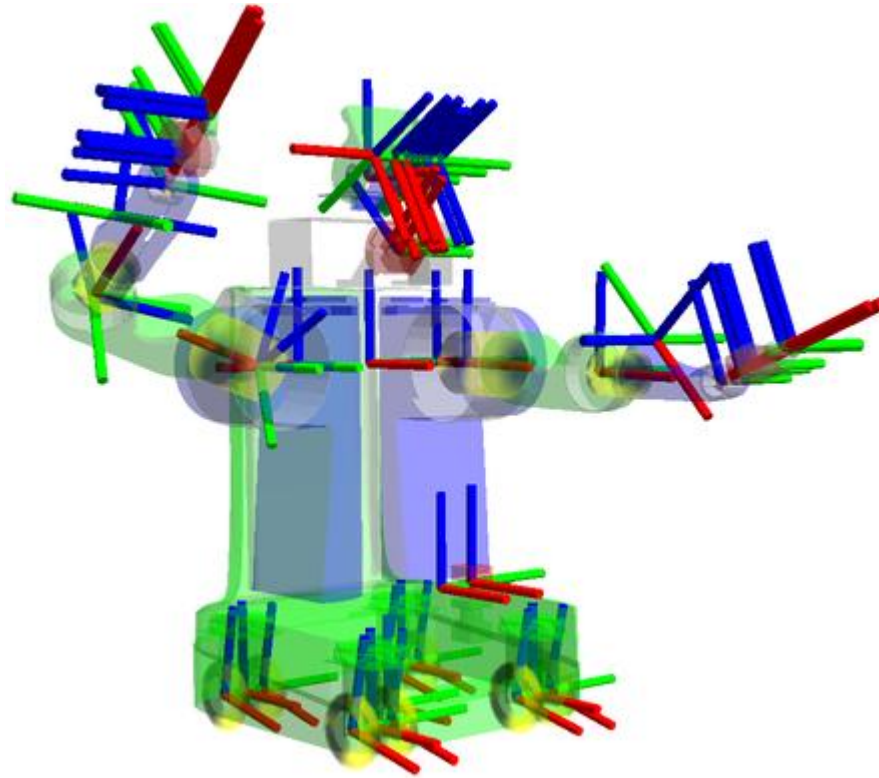


With respect to the relative coordinate frame:



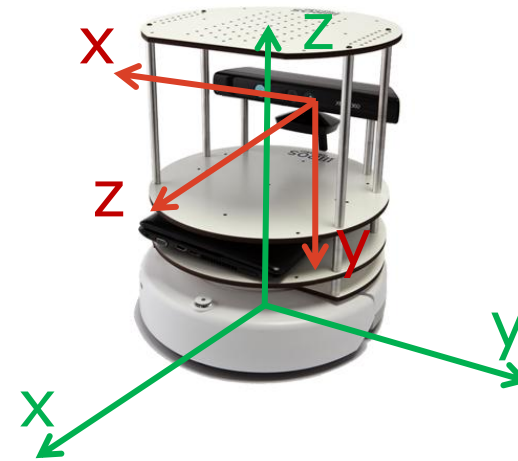
# Package TF in ROS

- Maintenance of the coordinate frames through time



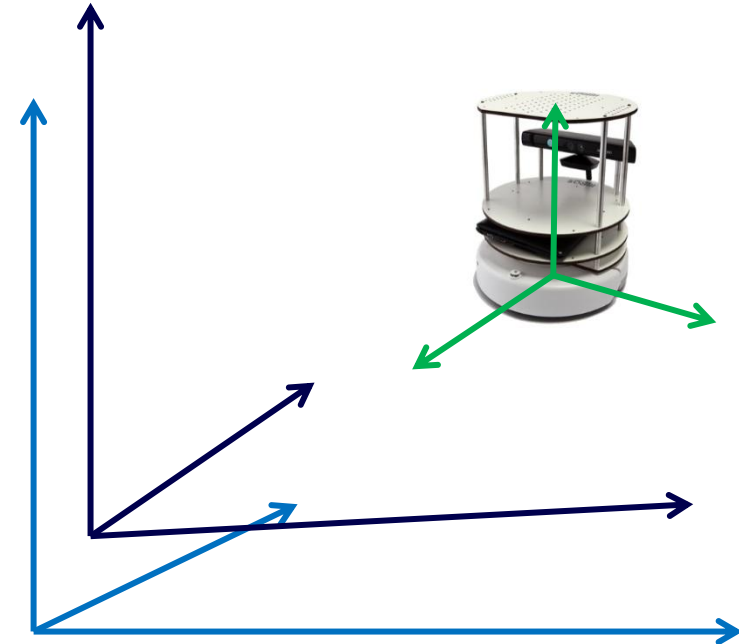
# Conventions

- Right-handed coordinate frame
- Orientation of the robot or object axes
  - x: forward
  - y: left
  - z: up
- Orientation of the camera axes
  - z: forward
  - x: right
  - y: down
- Rotation representations
  - quaternions
  - rotation matrix
  - rotations around X, Y and Z axes
  - Euler angles



# Coordinate frames on mobile platforms

- **map** (global map)
  - world coordinate frame
  - does not change (or very rarely)
  - long-term reference
  - useless in short-term
- **odom** (odometry)
  - world coordinate frame
  - changes with respect to odometry
  - useless in long-term
  - useful in short-term
- **base\_link** (robot)
  - attached to the robot
  - robot coordinate frame



# Tree of coordinate frames

- ROS TF2
  - tree of coordinate frames and their relative poses
  - distributed representation
  - dynamic representation
    - changes through time
  - accessible representation
    - querying relations between arbitrary coordinate frames

