Development of intelligent systems (RInS)

Transformations between coordinate frames

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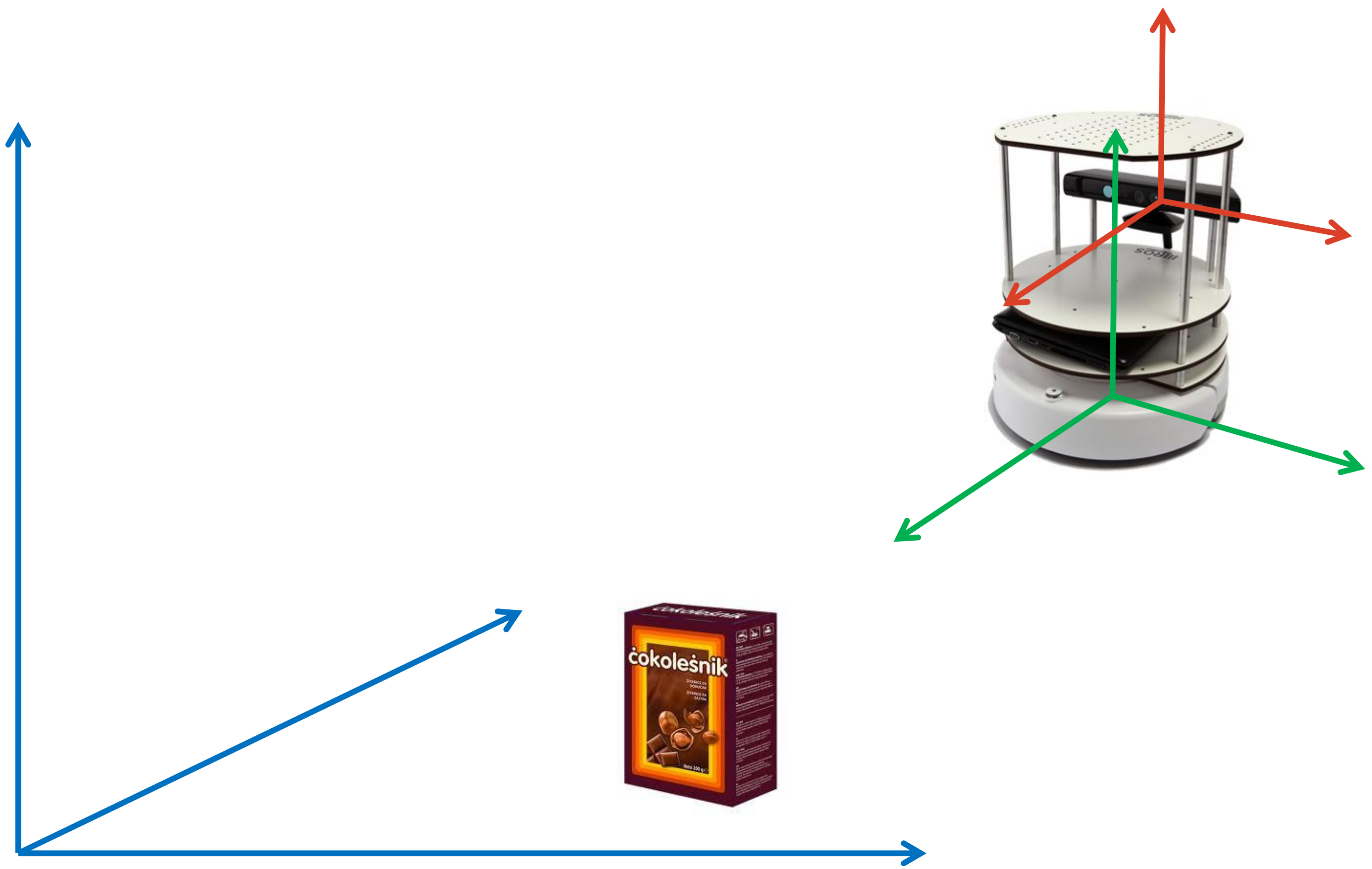
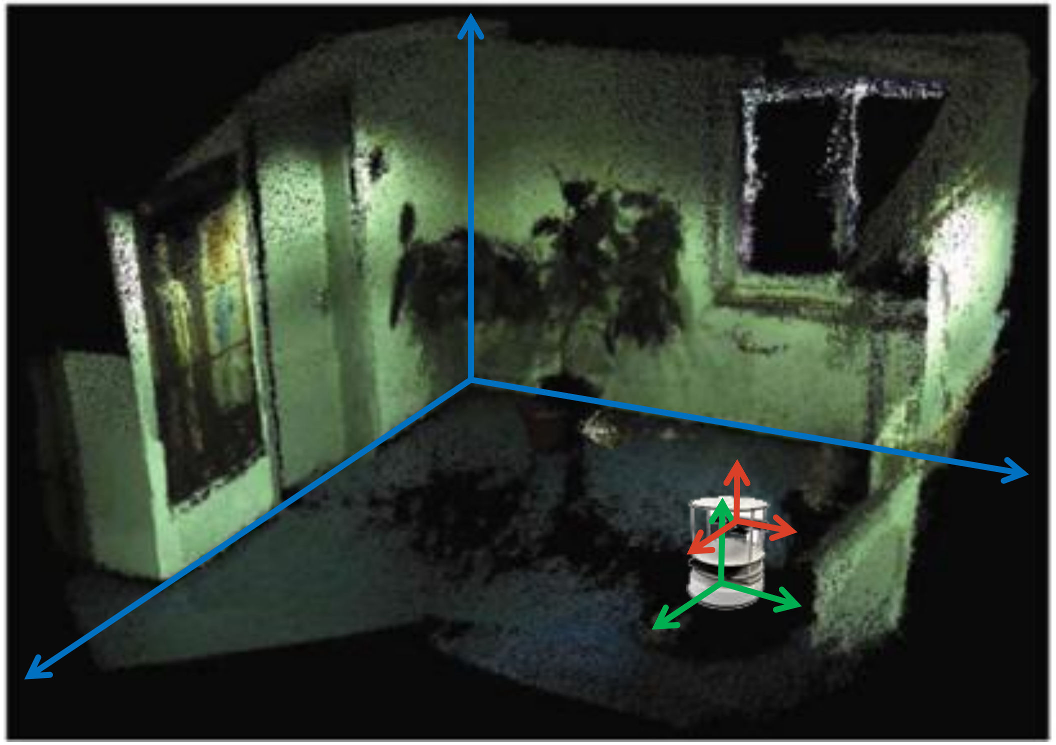
Literature: Tadej Bajd (2006).
Osnove robotike, chapter 2

Academic year: 2024/25

Coordinate frames



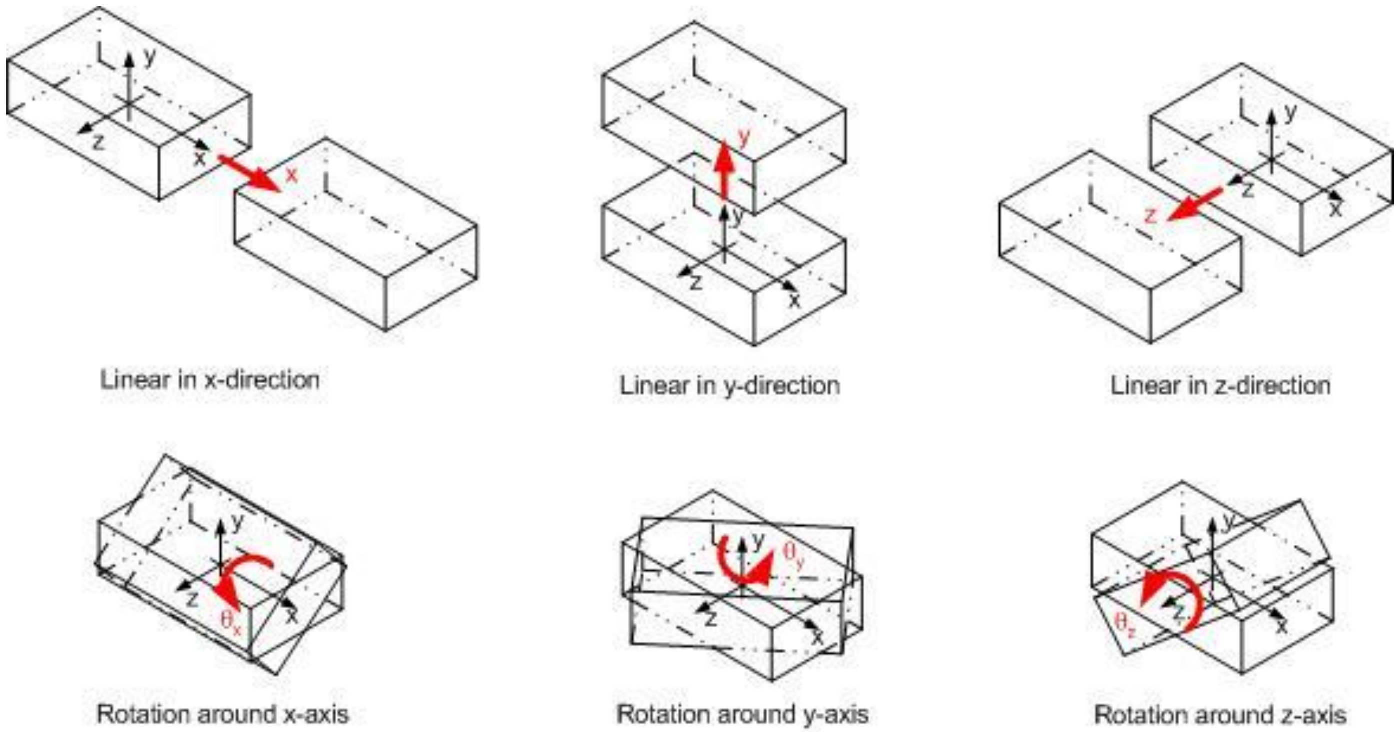
3D environment



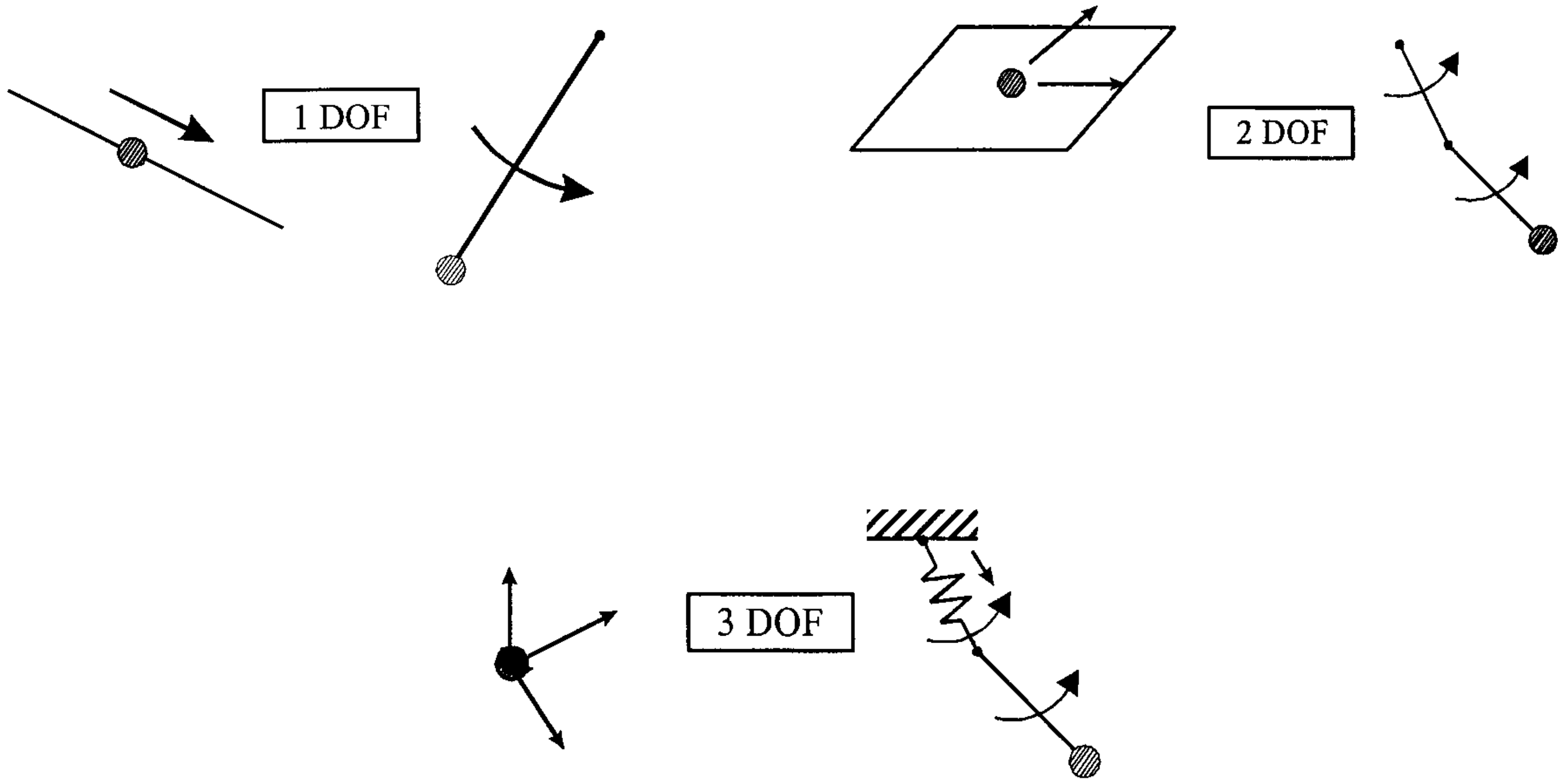
2D navigation

Degrees of freedom

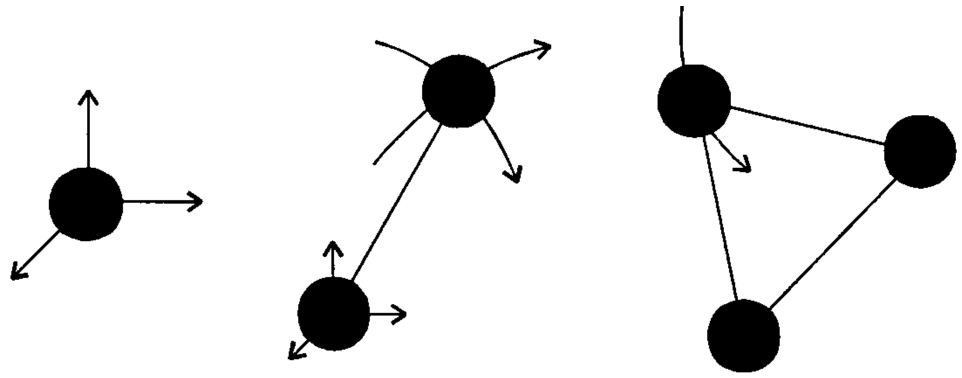
- DOF
- 6 DOF for full description of the pose of an object in space
  - 3 translations (position)
  - 3 rotations (orientation)



Degrees of freedom



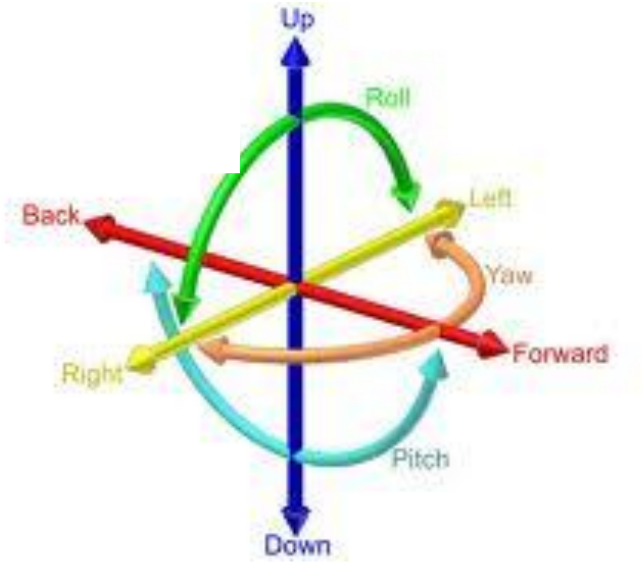
Degrees of freedom



3 translations 2 rotations 1 rotation

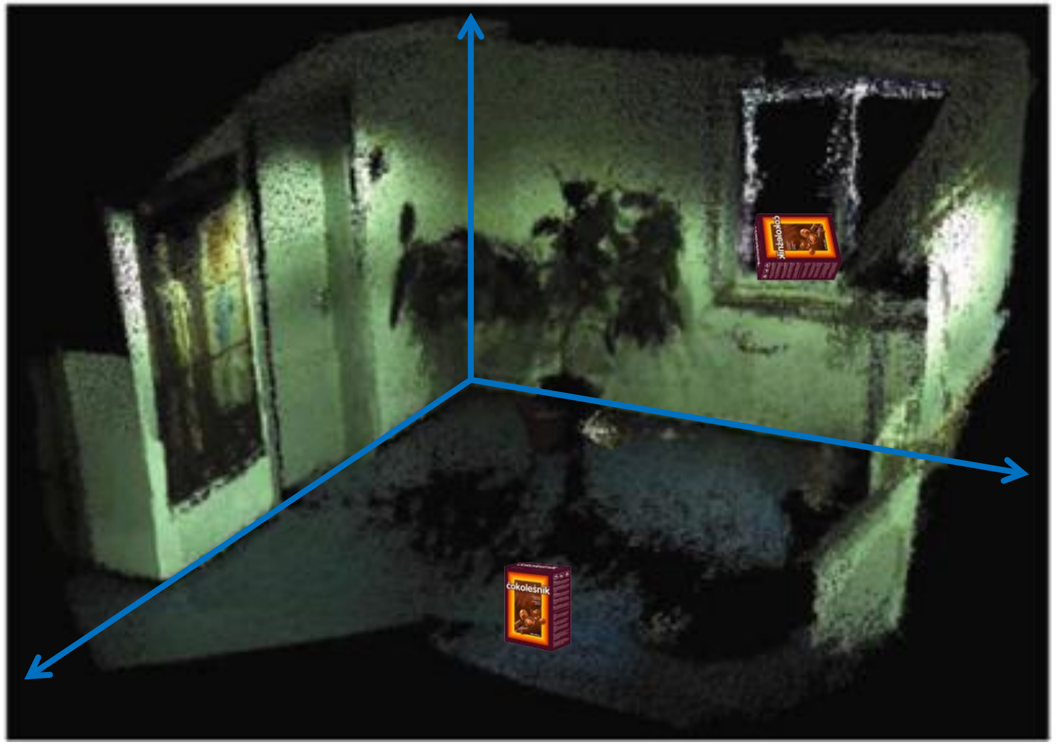
POSITION ORIENTATION

POSE



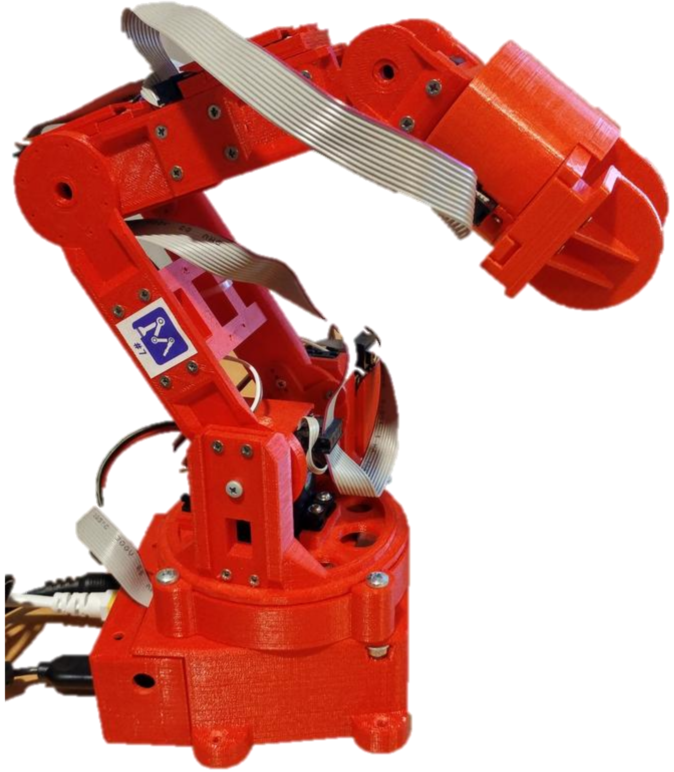
Position and orientation of the robot

Pose of the object in 3D space



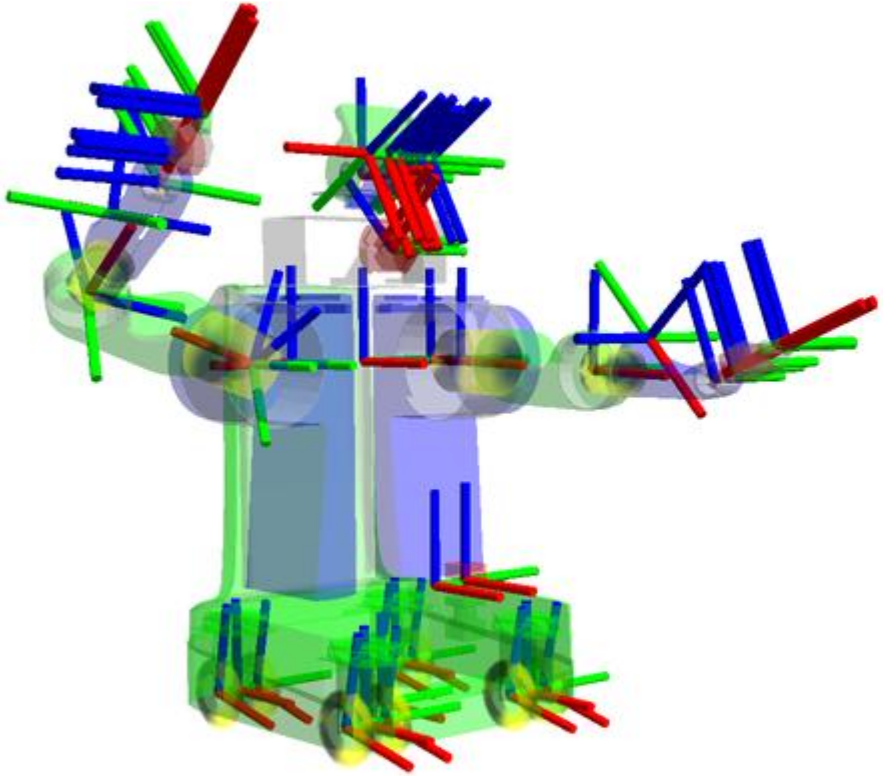
Robot manipulator

- ViCoS LCLWOS robot manipulator
  - 5DOF
- 6DOF needed for general grasping



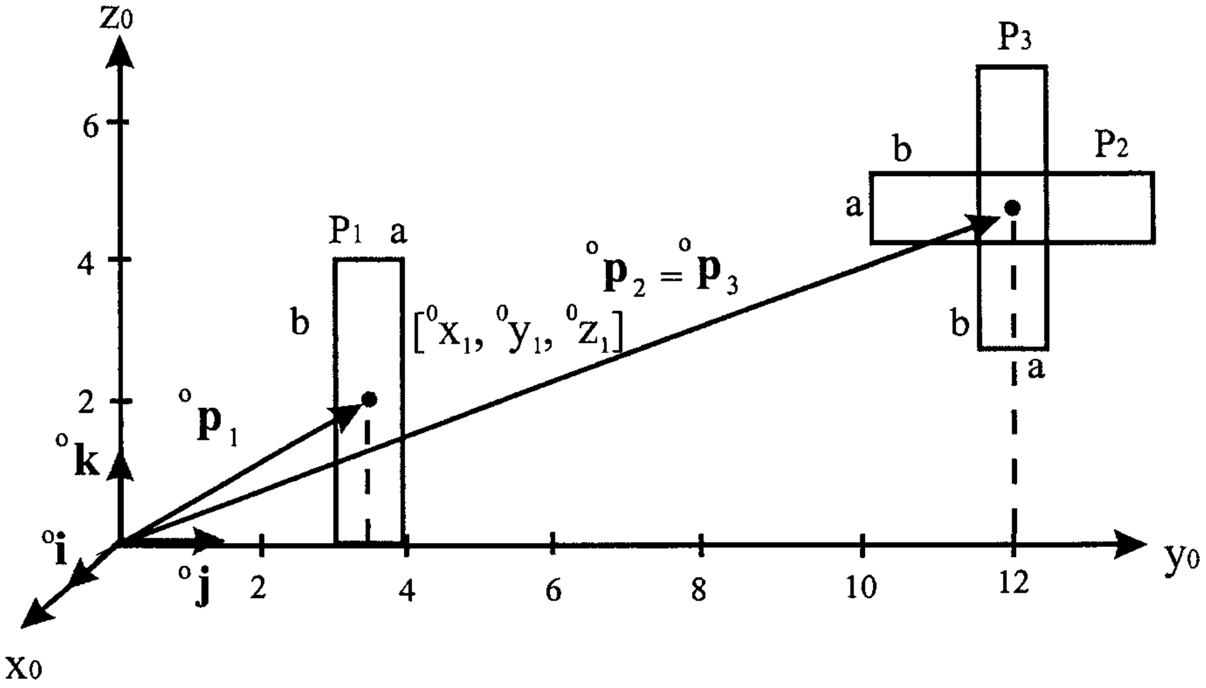
Chains of coordinate frames

- Transformations between coordinate frames



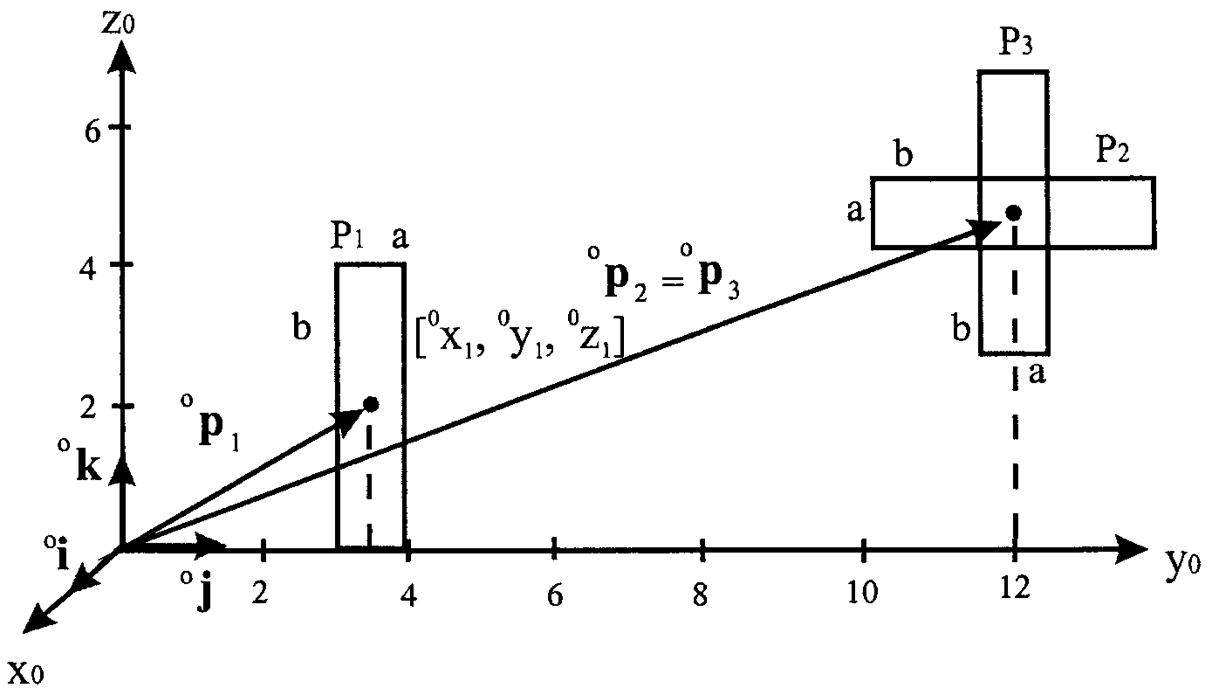
Position and orientation

- Pose=Position+Orientation
  - Position(P2)=Position (P3)
  - Position(P1)~=Position (P2)
  - Orientation(P1)=Orientation (P3)
  - Orientation(P2)~=Orientation (P3)
  - Pose(P1)~=Pose(P2)~=Pose(P3)



Translation and rotation

- Moving objects:
  - P1 v P3: Translation (T)
  - P2 v P3: Rotation (R)
  - P1 v P2: Translation in rotation

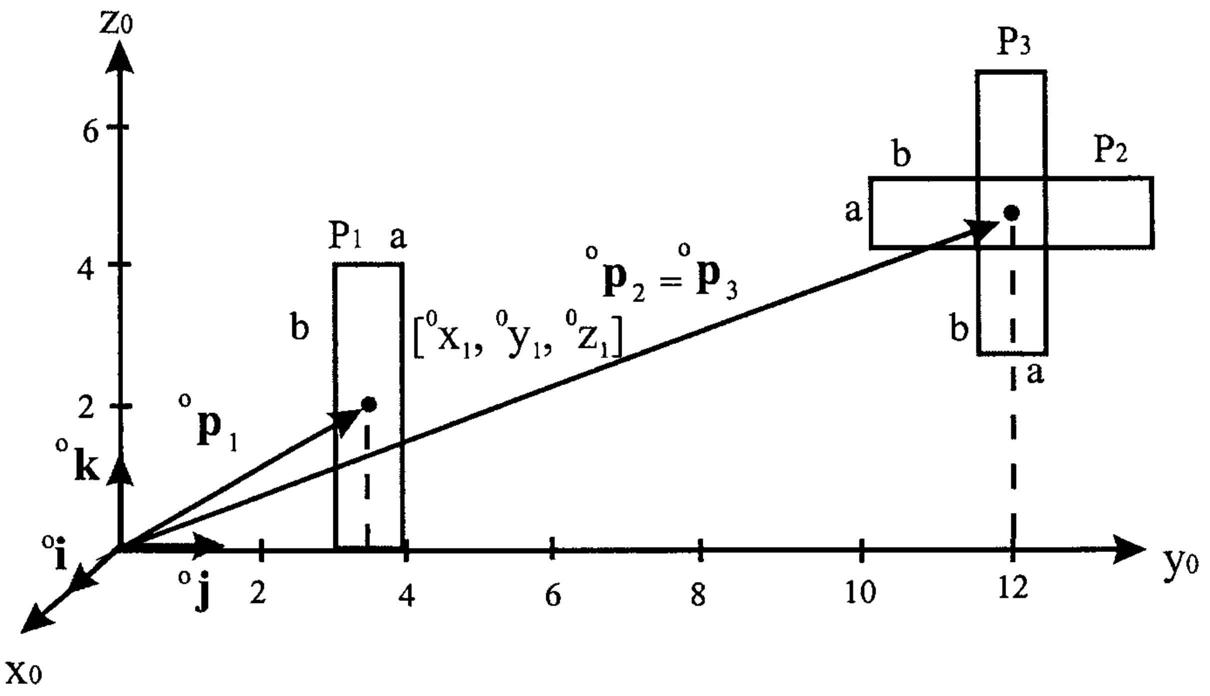


Position

- Position: vector from the origin of the coordinate frame to the point

- Position of the object P1:

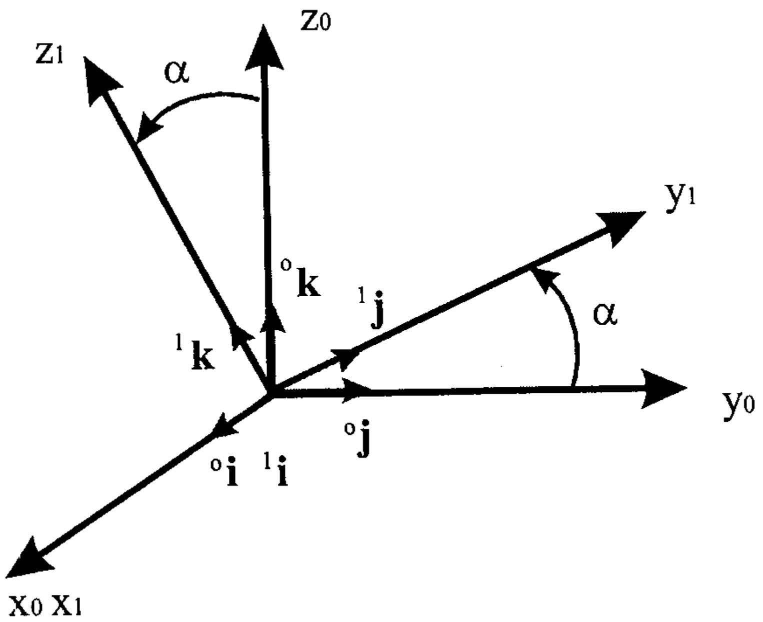
$$ {}^{0}\mathbf{p}_1 = {}^{0}x_1\,{}^{0}\mathbf{i} + {}^{0}y_1\,{}^{0}\mathbf{j} + {}^{0}z_1\,{}^{0}\mathbf{k} $$



Orientation

- Right-handed coordinate frame
- Rotation around x_0 axis:
- Rotation matrix:

$$ {}^0\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} $$



  - Orientation of c.f. O_1 with respect to c.f. O_0
  - Transformation of the vector ${}^1\mathbf{p}$ expressed in the c.f. O_1 into the coordinates expressed in the c.f. O_0 :

$$ {}^0\mathbf{p} = {}^0\mathbf{R}_1 \cdot {}^1\mathbf{p} $$

Rotation matrices

- Rotation around x axis:

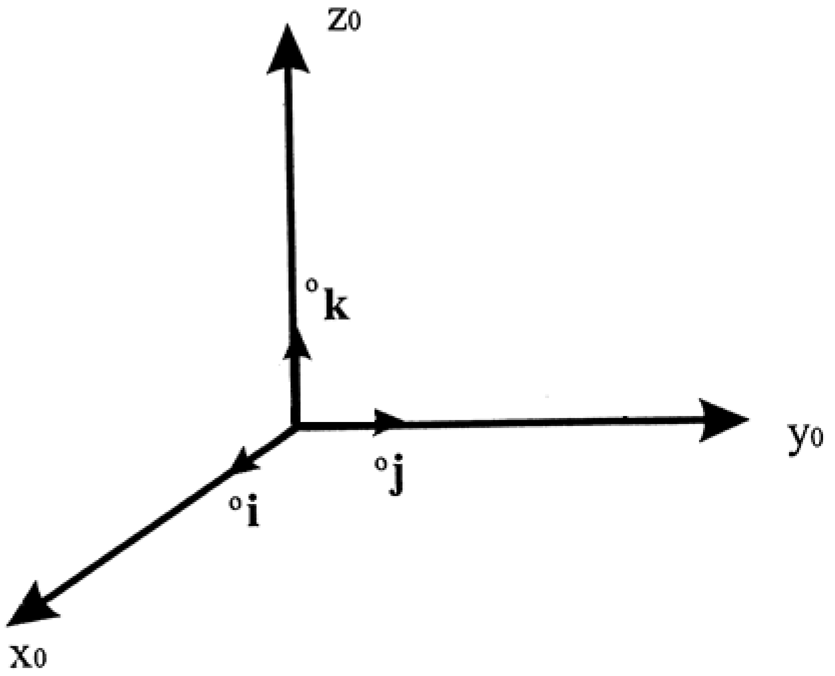
$$\mathbf{R}_{X,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

- Rotation around y axis :

$$\mathbf{R}_{Y,\alpha} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

- Rotation around z axis :

$$\mathbf{R}_{Z,\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Properties of rotation matrix

- Rotation is an orthogonal transformation matrix
- Inverse transformation:

$$ {}^{1}\mathbf{R}_0 = ({}^{0}\mathbf{R}_1)^{-1} = ({}^{0}\mathbf{R}_1)^{T} $$

- In the right-handed coordinate frame the determinant equals to 1

- Addition of angles:

$$ \mathbf{R}_{X,\alpha_1} \cdot \mathbf{R}_{X,\alpha_2} = \mathbf{R}_{X,\alpha_1+\alpha_2} $$

- Backward rotation:

$$ \mathbf{R}_{X,\alpha}{}^{-1} = \mathbf{R}_{X,-\alpha} $$

Consecutive rotations

- Postmultiplicate the vector with the rotation matrix

- Consecutive rotations:

$$ {}^{0}\mathbf{p} = {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{p} \qquad {}^{1}\mathbf{p} = {}^{1}\mathbf{R}_2 \cdot {}^{2}\mathbf{p} $$

$$ {}^{0}\mathbf{p} = {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{R}_2 \, {}^{2}\mathbf{p} $$

  - Rotation matrices are postmultiplicated:

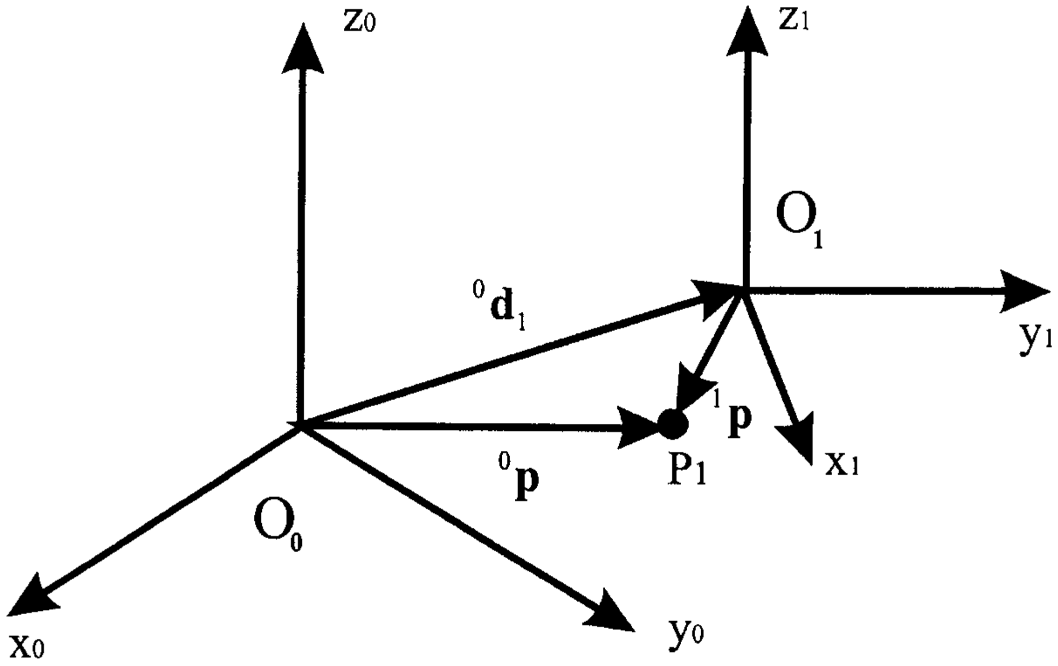
$$ {}^{0}\mathbf{R}_2 = {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{R}_2 $$

- In general:
  - Postmultiplicate matrices for all rotations
  - Rotations always refer to the respective relative current coordinate frame

$$ {}^{0}\mathbf{R}_n = {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{R}_2 \cdots {}^{n-1}\mathbf{R}_n $$

Transformations

- Transformation from one c.f. to another:



- If c.f. are parallel: $^{0}\mathbf{p} = {}^{1}\mathbf{p} + {}^{0}\mathbf{d}_1$
  - Only translation
- If c.f. are not parallel: $^{0}\mathbf{p} = {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{p} + {}^{0}\mathbf{d}_1$
  - Rotation and translation
  - General pose description

Matrix notation

- Three coordinate frames:

$$ {}^{0}\mathbf{p} = {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{p} + {}^{0}\mathbf{d}_1 \qquad {}^{1}\mathbf{p} = {}^{1}\mathbf{R}_2 \cdot {}^{2}\mathbf{p} + {}^{1}\mathbf{d}_2 $$

$$ {}^{0}\mathbf{p} = {}^{0}\mathbf{R}_2 \cdot {}^{2}\mathbf{p} + {}^{0}\mathbf{d}_2 $$

- Combine the transformations:

$$ {}^{0}\mathbf{R}_2 = {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{R}_2 \qquad {}^{0}\mathbf{d}_2 = {}^{0}\mathbf{d}_1 + {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{d}_2 $$

$$ {}^{0}\mathbf{p} = {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{R}_2\, {}^{2}\mathbf{p} + {}^{0}\mathbf{R}_1\, {}^{1}\mathbf{d}_2 + {}^{0}\mathbf{d}_1 $$

  - We can add the translation vectors if they are expressed in the same coordinate frame
  - The two equations in the matrix form:

$$ \begin{bmatrix} {}^{0}\mathbf{R}_1 & {}^{0}\mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} {}^{1}\mathbf{R}_2 & {}^{1}\mathbf{d}_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{R}_2 & {}^{0}\mathbf{R}_1 \cdot {}^{1}\mathbf{d}_2 + {}^{0}\mathbf{d}_1 \\ 0 & 1 \end{bmatrix} $$

Homogeneous transformations

- General pose

$$^{0}\mathbf{p} = \mathbf{R}\cdot {}^{1}\mathbf{p} + \mathbf{d}$$

can be expressed in the matrix form:

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Homogeneous transformation - homogenises (combines) rotation and translation in one matrix
- Very concise and convenient format

- Homogeneous matrix of size 4x4 (for 3D space)
  - One row is added, also 1 in the position vector

$$\begin{bmatrix} ^{0}\mathbf{p} \\ 1 \end{bmatrix} \quad \begin{bmatrix} ^{1}\mathbf{p} \\ 1 \end{bmatrix} \qquad \begin{bmatrix} ^{0}\mathbf{p} \\ 1 \end{bmatrix} = {}^{0}\mathbf{H}_{1} \begin{bmatrix} ^{1}\mathbf{p} \\ 1 \end{bmatrix}$$

Homogenous matrix

- Rotation R and translation d:

$$\left[\begin{array}{ccc|c} & & & \\ & R & & d \\ & & & \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

- Only rotation:

$$\left[\begin{array}{ccc|c} & & & 0 \\ & R & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Only translation:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Properties of homogeneous transformation

- Inverse of homogeneous transformation:

$$^{0}\mathbf{p} = \mathbf{R} \cdot {}^{1}\mathbf{p} + \mathbf{d}$$

$$^{1}\mathbf{p} = \mathbf{R}^{T} \cdot {}^{0}\mathbf{p} - \mathbf{R}^{T}\mathbf{d}$$

$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{R}^{T} & -\mathbf{R}^{T} \cdot \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Consecutive poses:
  - Postmultiplication of homogeneous transformations:

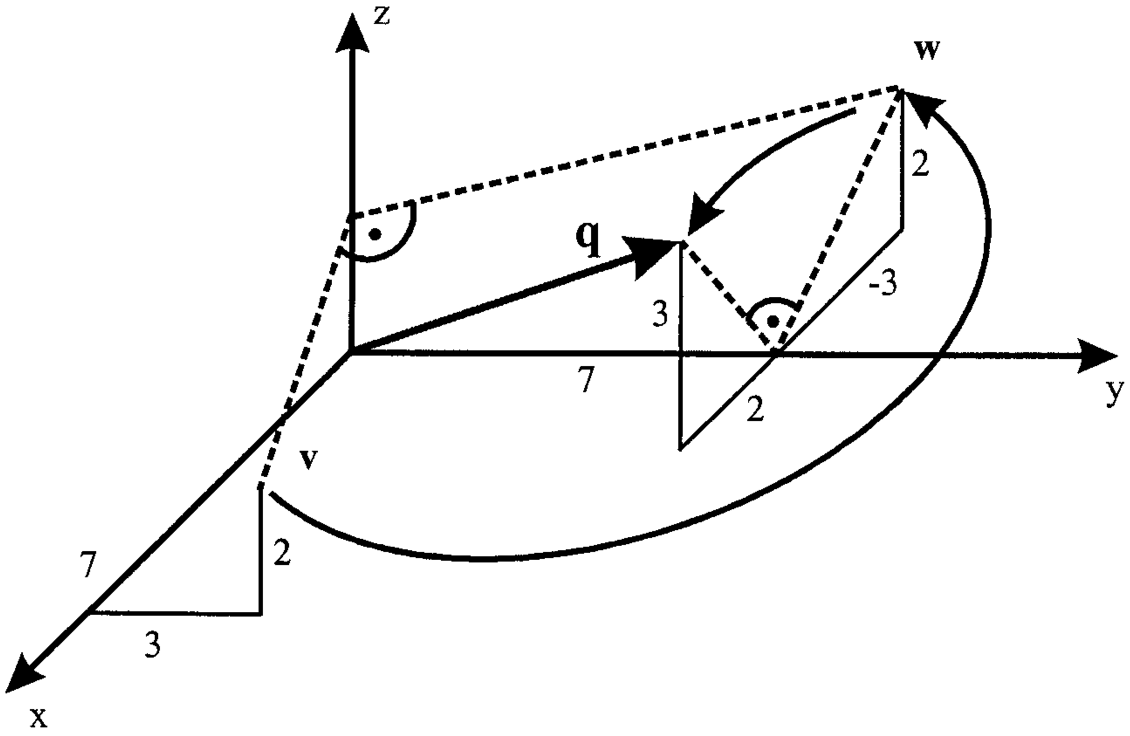
$$^{0}\mathbf{H}_{2} = {}^{0}\mathbf{H}_{1} \cdot {}^{1}\mathbf{H}_{2}$$

$$^{0}\mathbf{H}_{n} = {}^{0}\mathbf{H}_{1} \cdot {}^{1}\mathbf{H}_{2} \ldots {}^{n-1}\mathbf{H}_{n}$$

  - An element can be transformed arbitrary number of times – by multiplying homogeneous matrices

Example

- Two rotations
  - Vector $\mathbf{v} = [7, 3, 2, 1]^T$
    first rotate for 90° around z axis $\mathbf{w} = \mathbf{Rot}(z, 90)\,\mathbf{v}$
    and then for 90° around y axis $\mathbf{q} = \mathbf{Rot}(y, 90)\,\mathbf{w}$



Example– two rotations

$$\mathbf{w} = \mathbf{Rot}\,(z, 90)\ \mathbf{v}$$
$$\mathbf{q} = \mathbf{Rot}\,(y,90)\ \mathbf{w}$$

$$\mathbf{q} = \ \mathbf{Rot}\,(y,90)\ \mathbf{Rot}\,(z, 90)\ \mathbf{v}$$

$$\mathbf{Rot}(y,90)\ \mathbf{Rot}(z, 90) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 1 \end{bmatrix}$$

Example - translation

- After two rotations also translate the vector for (4,-3,7)
  - Merge
    - Translation with rotations $\mathbf{Trans}(4\mathbf{i} \ -3\mathbf{j} + 7\mathbf{k})$ $\mathbf{Rot}(y,90) \cdot \mathbf{Rot}(z, 90)$

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \mathbf{Trans}\,(4, -3, 7)\ \mathbf{Rot}\,(y,90)\ \mathbf{Rot}\,(z, 90)$$

- Transformation of the point (7,3,2):

$$\mathbf{x} = \mathbf{H}_1 \cdot \mathbf{v} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

Transformation of the coordinate frame

- Homogeneous transformation matrix transforms the base coordinate frame

$$\mathbf{Trans}(4, -3, 7)\ \mathbf{Rot}(y,90)\ \mathbf{Rot}(z, 90)$$

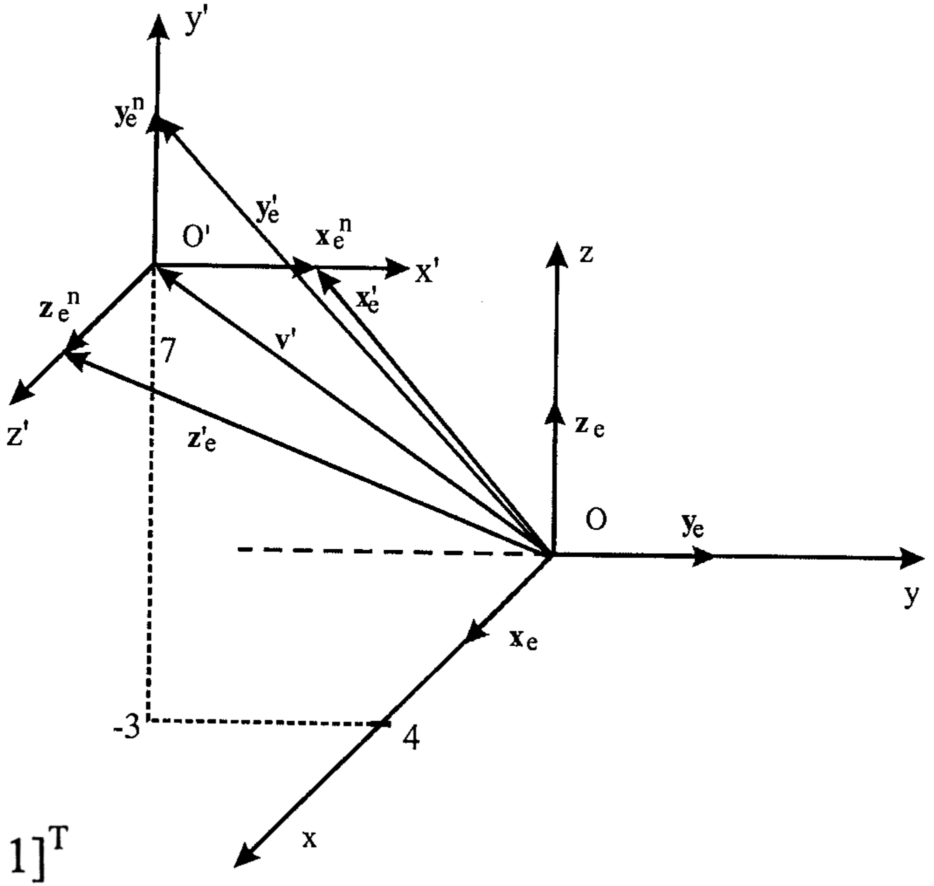
  - Vector of origin of c.f.:

$$\mathbf{H}_1 \cdot \mathbf{v} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 7 \\ 1 \end{bmatrix} = \mathbf{v}'$$

  - Unit vectors:

$$\begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 7 \\ 1 \end{bmatrix} = \mathbf{x}_e'$$

$$\mathbf{y}_e' = [4, -3, 8, 1]^T \quad , \quad \mathbf{z}_e' = [5, -3, 7, 1]^T$$



Pose of the coordinate frame

- Unit vectors of the new coordinate frame:

$$\mathbf{x}_e^n: \quad 4\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} - 4\mathbf{i} + 3\mathbf{j} - 7\mathbf{k} = 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{x}_e^n = [0, 1, 0, 0]^T$$

$$\mathbf{y}_e^n: 4\mathbf{i} - 3\mathbf{j} + 8\mathbf{k} - 4\mathbf{i} + 3\mathbf{j} - 7\mathbf{k} = 0\mathbf{i} + 0\,\mathbf{j} + 1\mathbf{k}$$

$$\mathbf{y}_e^n = [0, 0, 1, 0]^T$$

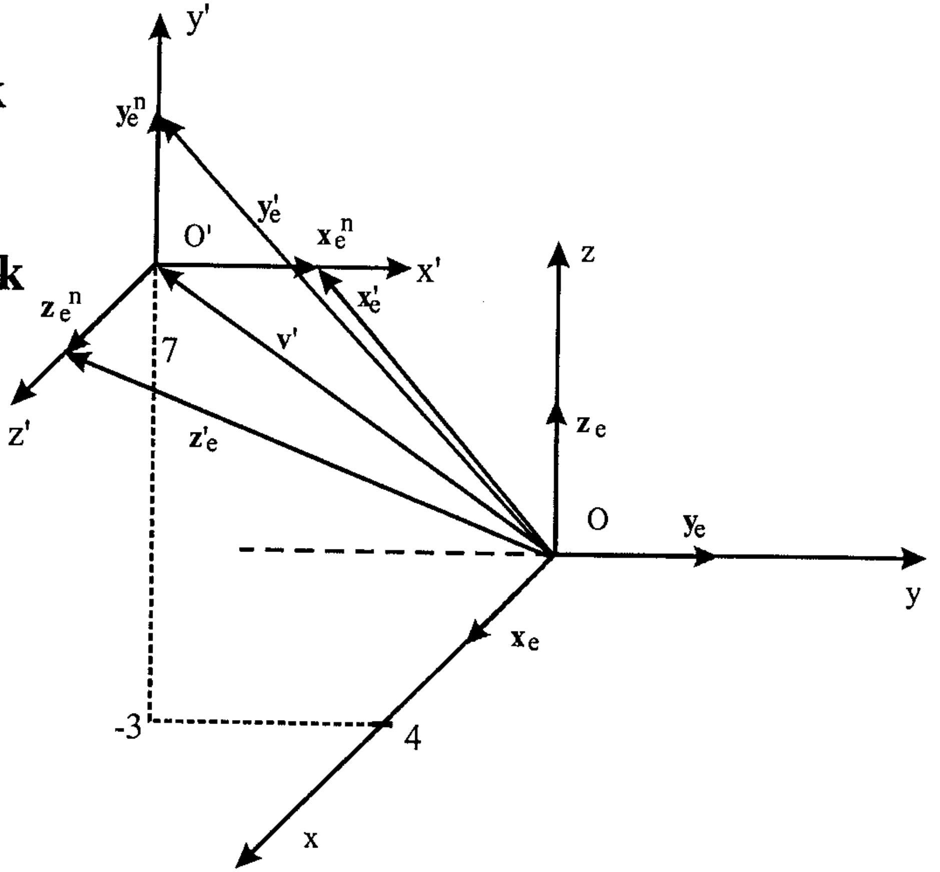
$$\mathbf{z}_e^n = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} - 4\mathbf{i} + 3\mathbf{j} - 7\mathbf{k} = 1\mathbf{i} + 0\,\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{z}_e^n = [1, 0, 0, 0]^T$$

- Transformaction matrix descibes the coordinate frame!

$$\begin{array}{c} \begin{array}{cccc} \mathbf{x}_e^n & \mathbf{y}_e^n & \mathbf{z}_e^n & \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\mathbf{v}' = [4, -3, 7, 1]^T$$



Movement of the coordinate frame

- Premultiplication or postmultiplication (of an object or c.f.) with transformation
- Example:
  - Coordinate frame:

$$\mathbf{C} = \begin{array}{c@{}c} \begin{array}{cccc} \mathbf{i}_c & \mathbf{j}_c & \mathbf{k}_c & \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \\ \end{array} \end{array}$$

  - Transformation:

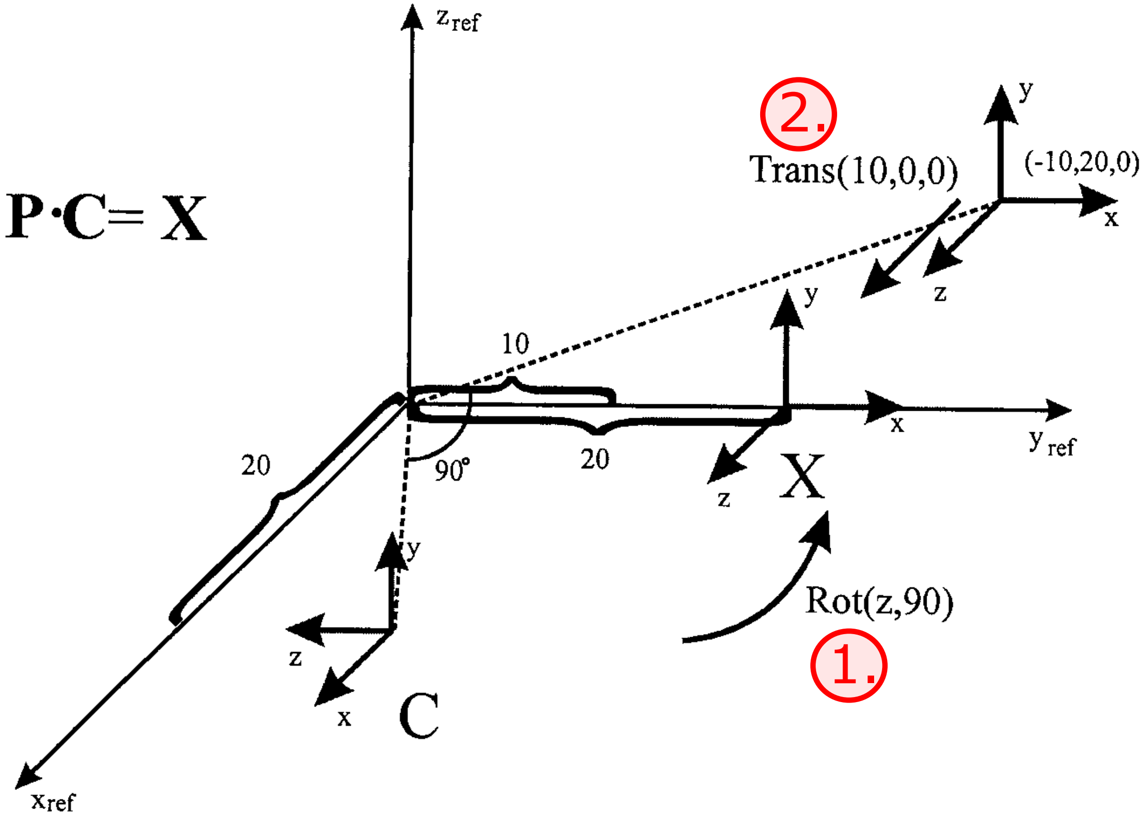
$$\mathbf{P} = \begin{bmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{Trans}(10,0,0) \cdot \mathbf{Rot}(z,90)$$

Premultiplication

$$\mathbf{P}\cdot\underline{\mathbf{C}}=\mathbf{X}=\begin{bmatrix}0 & -1 & 0 & 10\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0 & 20\\ 0 & 0 & -1 & 10\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 20\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

- The pose of the object is transformed with respect to the **fixed reference** coordinate frame in which the object coordinates were given.
- Order of transformations:

$$\mathbf{Trans}(10,0,0)\cdot\mathbf{Rot}(z,90)$$

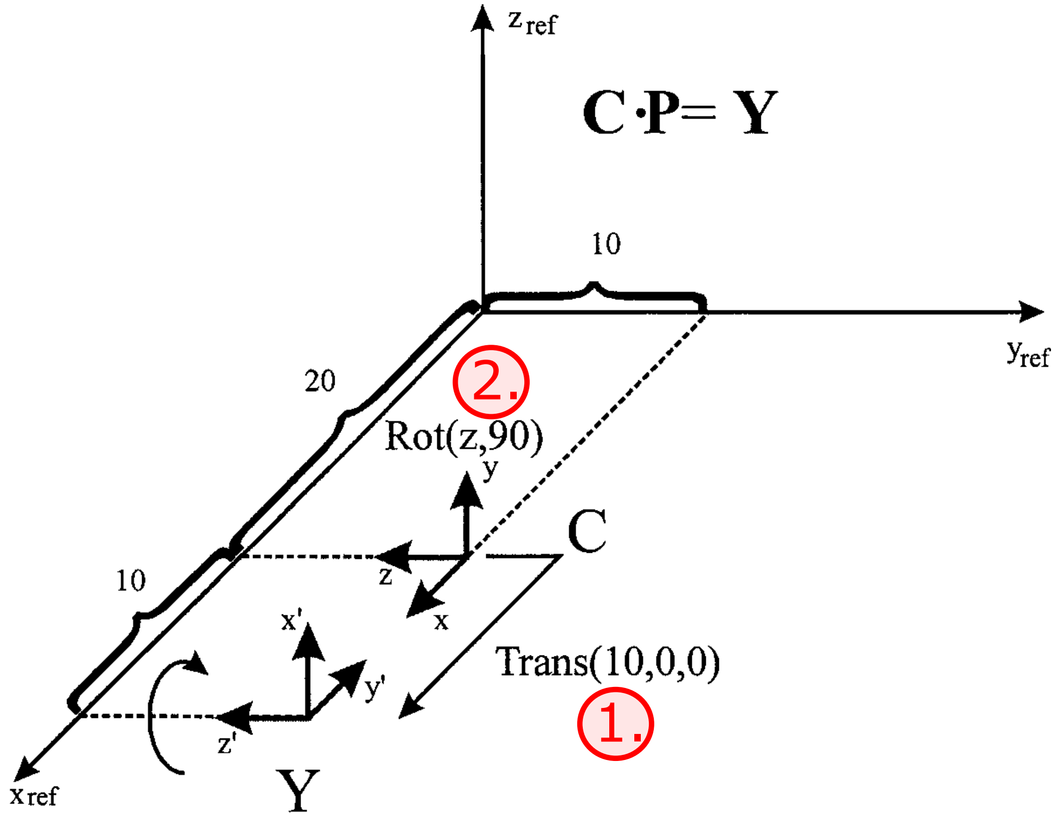


Postmultiplication

$$C \cdot P = Y = \begin{array}{c} \begin{array}{cccc} \mathbf{i}_c & \mathbf{j}_c & \mathbf{k}_c & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \\ \end{array} \cdot \begin{bmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 30 \\ 0 & 0 & -1 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The pose of the object is transformed with respect to its **own relative current** coordinate frame
- Order of transformations:

$$\mathbf{Trans}(10,0,0) \cdot \mathbf{Rot}(z,90)$$



Movement of the reference c.f.

- Example: **Trans**(2,1,0)**Rot**(z,90)

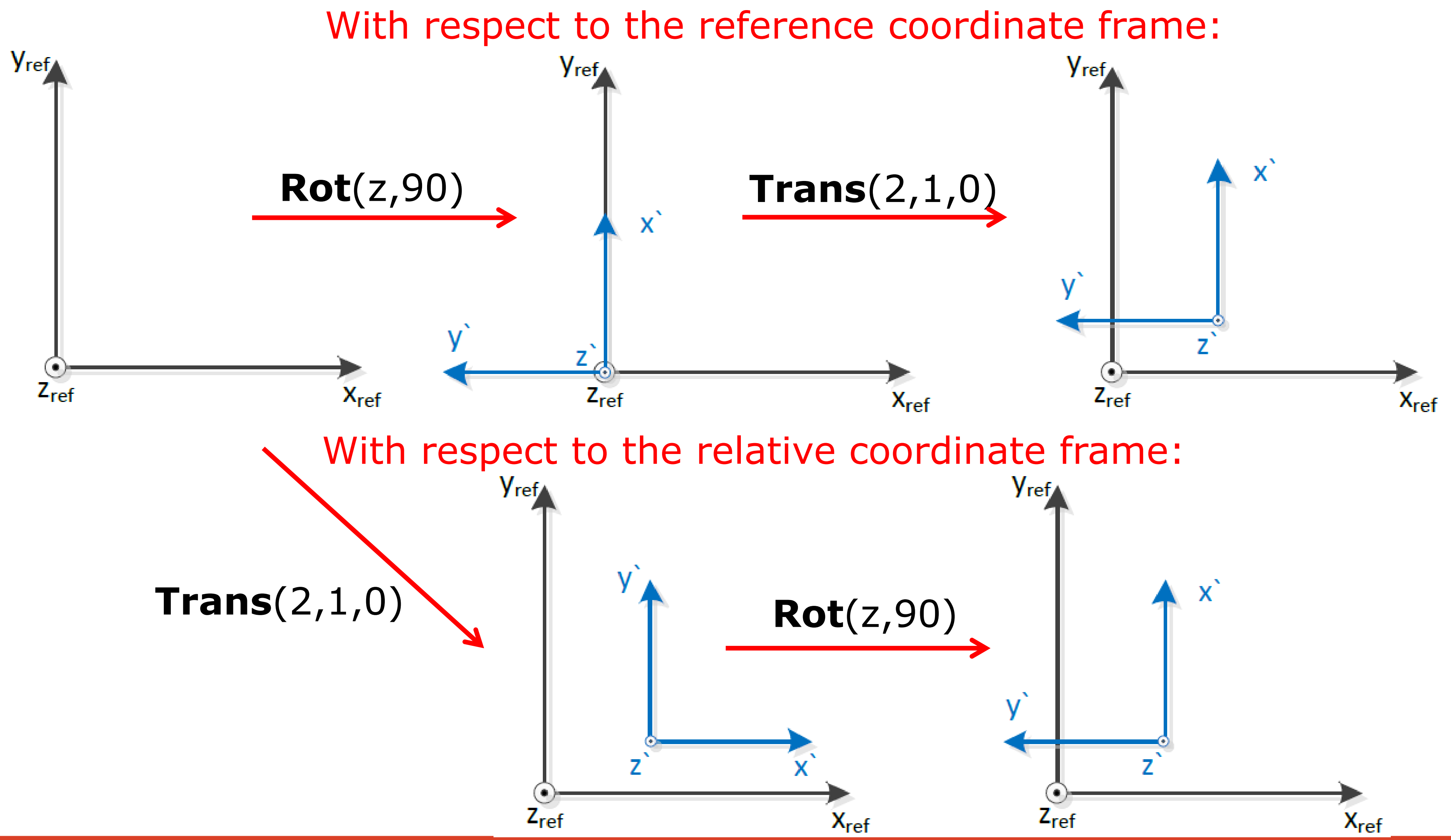
$$\begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

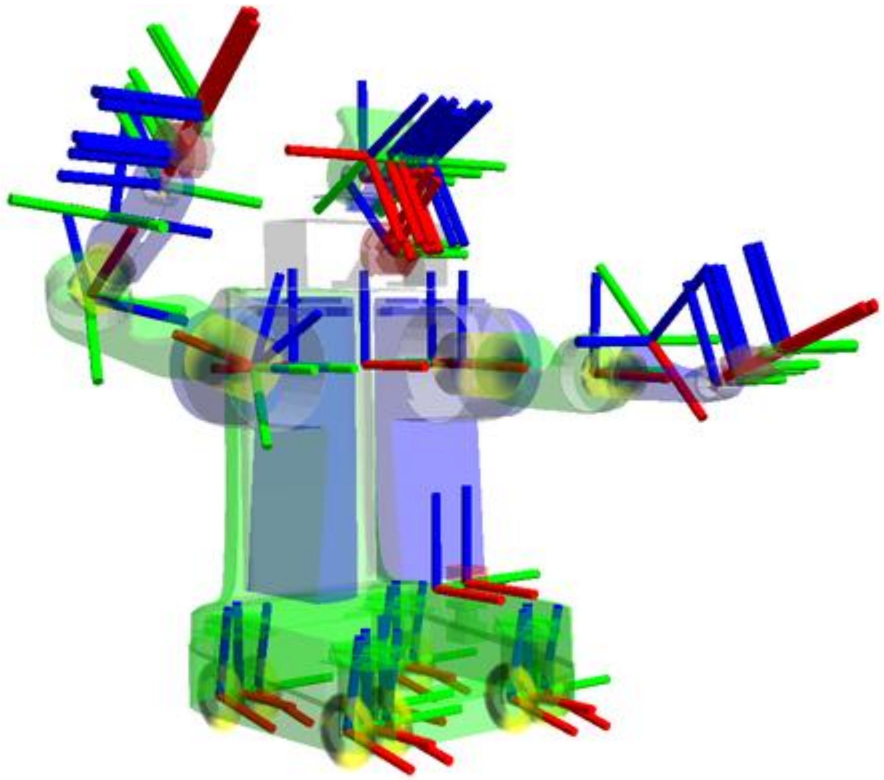
Movement of the reference c.f.

- Example: **Trans**(2,1,0)**Rot**(z,90)



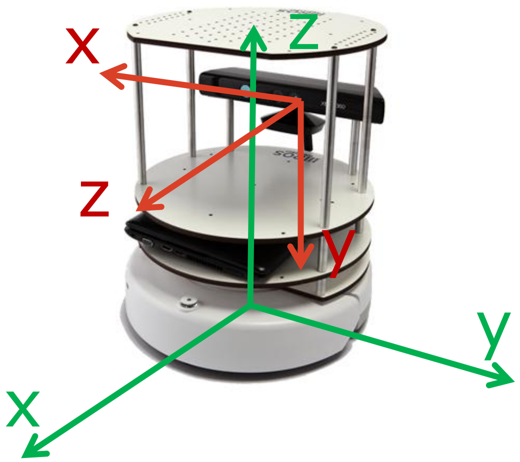
Package TF in ROS

- Maintenance of the coordinate frames through time



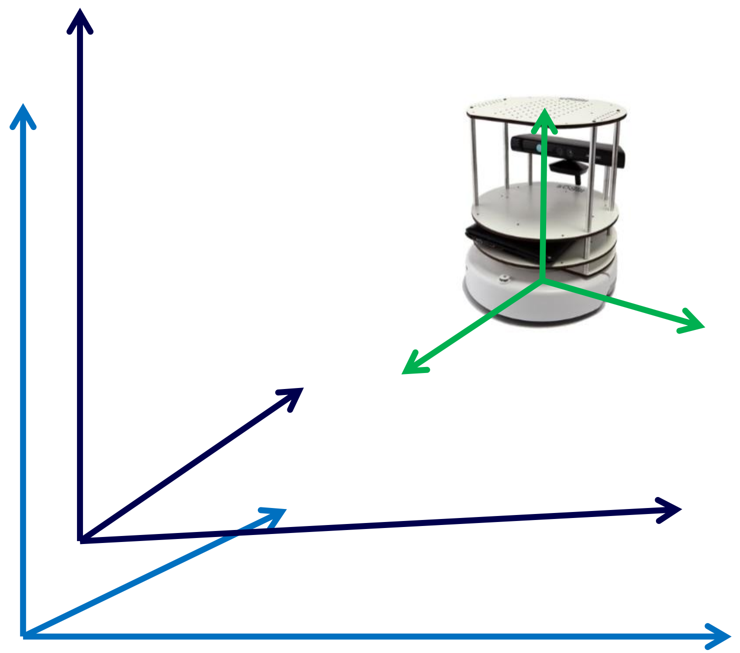
Conventions

- Right-handed coordinate frame
- Orientation of the robot or object axes
  - x: forward
  - y: left
  - z: up
- Orientation of the camera axes
  - z: forward
  - x: right
  - y: down
- Rotation representations
  - quaternions
  - rotation matrix
  - rotations around X, Y and Z axes
  - Euler angles



Coordinate frames on mobile plaforms

- map (global map)
  - world coordinate frame
  - does not change (or very rarely)
  - long-term reference
  - useless in short-term
- odom (odometry)
  - world coordinate frame
  - changes with respect to odometry
  - useless in long-term
  - uselful in short-term
- base_link (robot)
  - attached to the robot
  - robot coordinate frame



Tree of coordinate frames

- ROS TF2
  - tree of coordinate frames and their relative poses
  - distributed representation
  - dynamic representation
    - changes through time
  - accessible representation
    - querying relations between arbitrary coordinate frames