1 Master method

Using the master method solve the following recurrences.

$$T(n) = 2 \cdot T(\frac{n}{4}) + 1 \tag{1}$$

$$T(n) = 2 \cdot T(\frac{n}{4}) + \sqrt{n} \tag{2}$$

$$T(n) = 2 \cdot T(\frac{n}{4}) + n \tag{3}$$

$$T(n) = 2 \cdot T(\frac{n}{4}) + n^2 \tag{4}$$

$$T(n) = 7 \cdot T(\frac{n}{2}) + n^2 \tag{5}$$

$$T(n) = 4 \cdot T(\frac{n}{2}) + n^2 lg(n) \tag{6}$$

2 Akra-Bazzi

Find tight asymptotic bounds of the following recursive functions.

$$T(n) = 2 \cdot T\Big(\frac{n}{2}\Big) + n \tag{7}$$

$$T(n) = 2 \cdot T\Big(\frac{n}{2}\Big) + n \cdot lg(n) \tag{8}$$

$$T(n) = T\Big(\left\lceil \frac{n}{2} \right\rceil\Big) + T\Big(\left\lfloor \frac{n}{2} \right\rfloor\Big) + n \tag{9}$$

$$T(n) = 2 \cdot T\Big(\frac{n}{2}\Big) + \frac{9}{2} \cdot T\Big(\frac{n}{3}\Big) + \theta(n) \tag{10}$$

$$T(n) = 2 \cdot T\Big(\frac{n}{4}\Big) + 3 \cdot T\Big(\frac{n}{6}\Big) + n \cdot lg(n) \tag{11}$$

3 Annihilators

For each of the following recurrences find the closed form solution, estimate upper asymptotic bound and prove your solution is correct.

$$T(n) = T(n-1) + 1; T(0) = 0 \tag{12}$$

$$T(n) = T(n-1) + n; T(1) = 1 \tag{13}$$

$$T(n) = T(n-1) + 2 \cdot n + 1; T(0) = 0 \tag{14}$$

$$T(n) = T(n-1) + \binom{n}{2}; T(0) = 0 \tag{15}$$

$$T(n) = T(n-1) + 2^n; T(0) = 0 \tag{16}$$

$$T(n) = 3 \cdot T(n-1); T(0) = 1 \tag{17}$$

$$T(n) = 2 \cdot T(n-1) + 1; T(0) = 0 \tag{18}$$

$$T(n) = T(n-1) + T(n-2) + 1; T(-1) = 0, T(0) = 1 \tag{19}$$

Steps on solving linear recurrences.

- Write the recurrence in operator form
- Extract an annihilator for the recurrence
- Factor the annihilator (if necessary)
- Extract the generic solution from the annihilator
- Solve for coefficients using base cases (if known)

Operator	Definition
addition	$(f+g)(n) := f(n)+g(n)$
subtraction	$(f-g)(n) := f(n)-g(n)$
multiplication	$(\alpha \cdot f)(n) := \alpha \cdot (f(n))$
shift	$\boldsymbol{E} f(n) := f(n+1)$
k-fold shift	$\boldsymbol{E}^k f(n) := f(n+k)$
composition	$(\boldsymbol{X}+\boldsymbol{Y})f := \boldsymbol{X}f + \boldsymbol{Y}f$
	$(\boldsymbol{X}-\boldsymbol{Y})f := \boldsymbol{X}f - \boldsymbol{Y}f$
	$\boldsymbol{XY}f := \boldsymbol{X}(\boldsymbol{Y}f) = \boldsymbol{Y}(\boldsymbol{X}f)$
distribution	$\boldsymbol{X}(f+g) = \boldsymbol{X}f + \boldsymbol{X}g$

Operator	Functions annihilated	
$\boldsymbol{E}-1$	α	
$\boldsymbol{E}-a$	αa^n	
$(\boldsymbol{E}-a)(\boldsymbol{E}-b)$	$\alpha a^n + \beta b^n$	[if $a \neq b$]
$(\boldsymbol{E}-a_0)(\boldsymbol{E}-a_1)\cdots(\boldsymbol{E}-a_k)$	$\sum_{i=0}^k \alpha_i a_i^n$	[if a_i distinct]
$(\boldsymbol{E}-1)^2$	$\alpha n + \beta$	
$(\boldsymbol{E}-a)^2$	$(\alpha n + \beta)a^n$	
$(\boldsymbol{E}-a)^2(\boldsymbol{E}-b)$	$(\alpha n + \beta)a^n + \gamma b^n$	[if $a \neq b$]
$(\boldsymbol{E}-a)^d$	$(\sum_{i=0}^{d-1} \alpha_i n^i)a^n$	

If $\boldsymbol{X}$ annihilates f, then $\boldsymbol{X}$ also annihilates $\boldsymbol{E}f$.
If $\boldsymbol{X}$ annihilates both f and g, then $\boldsymbol{X}$ also annihilates $f \pm g$.
If $\boldsymbol{X}$ annihilates f, then $\boldsymbol{X}$ also annihilates αf, for any constant α.

If $\boldsymbol{X}$ annihilates f and $\boldsymbol{Y}$ annihilates g, then $\boldsymbol{XY}$ annihilates $f \pm g$.