Computational topology

Lab work, 6^{th} week

1. Let $S = \{A(0,0), B(0.5,0.5), C(0,1), D(1,2), E(1.5,1.5), F(2,1.5), G(2.5,1), H(2,0)\} \subset \mathbb{R}^2$. Build the Vietoris-Rips complex $\mathrm{Rips}(S,R)$ for

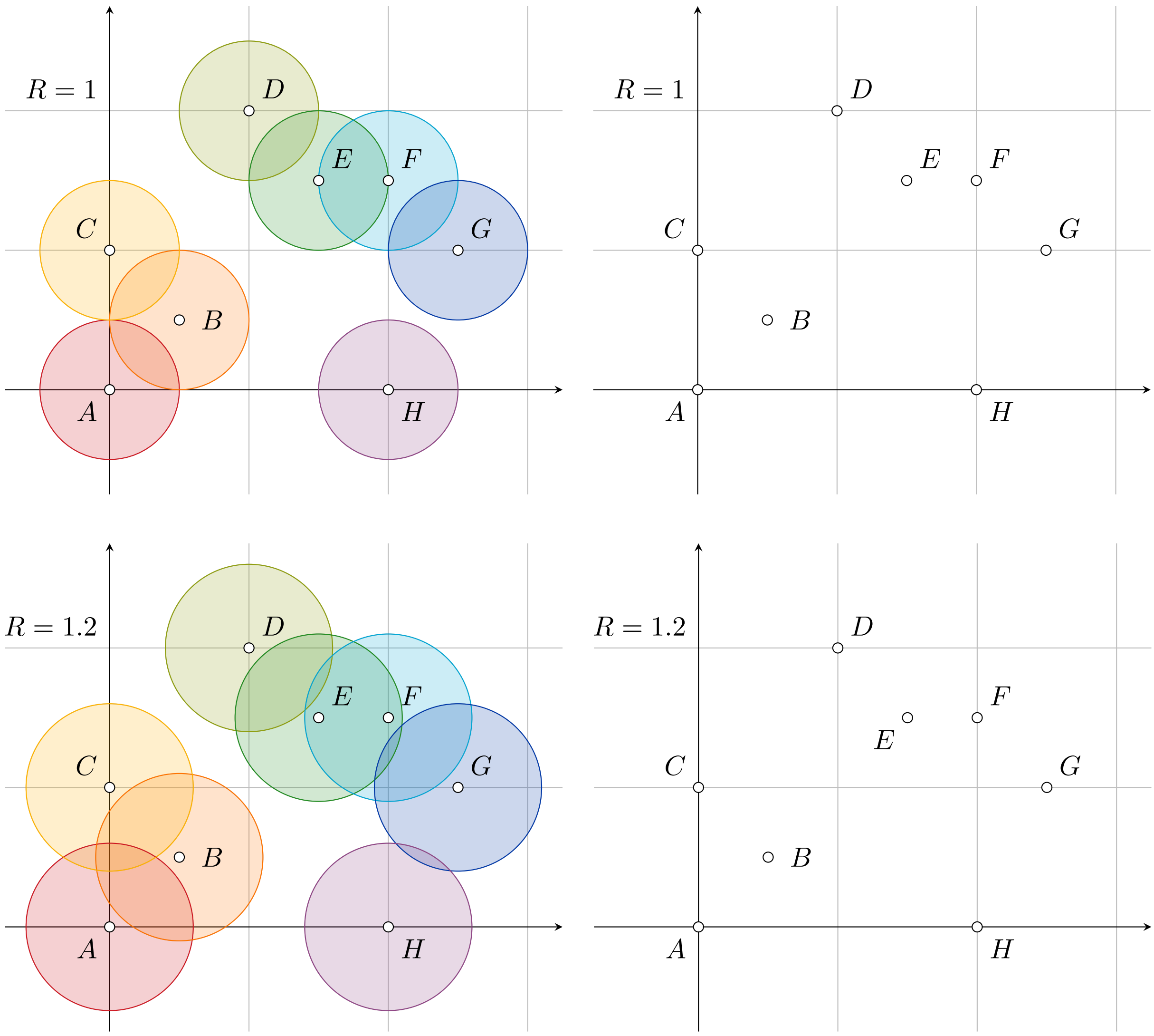
   (a) $R = 1$,

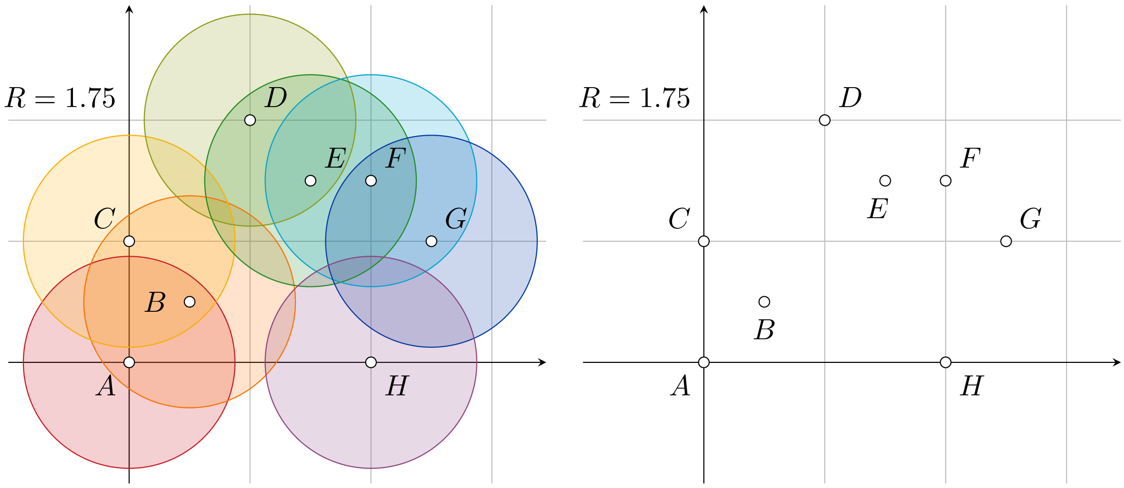
   (b) $R = 1.2$,

   (c) $R = 1.75$.

   In each case list all the simplices and determine its dimension.

   Assuming there is a sensor placed at each point of S and all sensors can detect points that are at distance 1.75 or less, is the area covered by the sensors connected? Does it contain any holes?





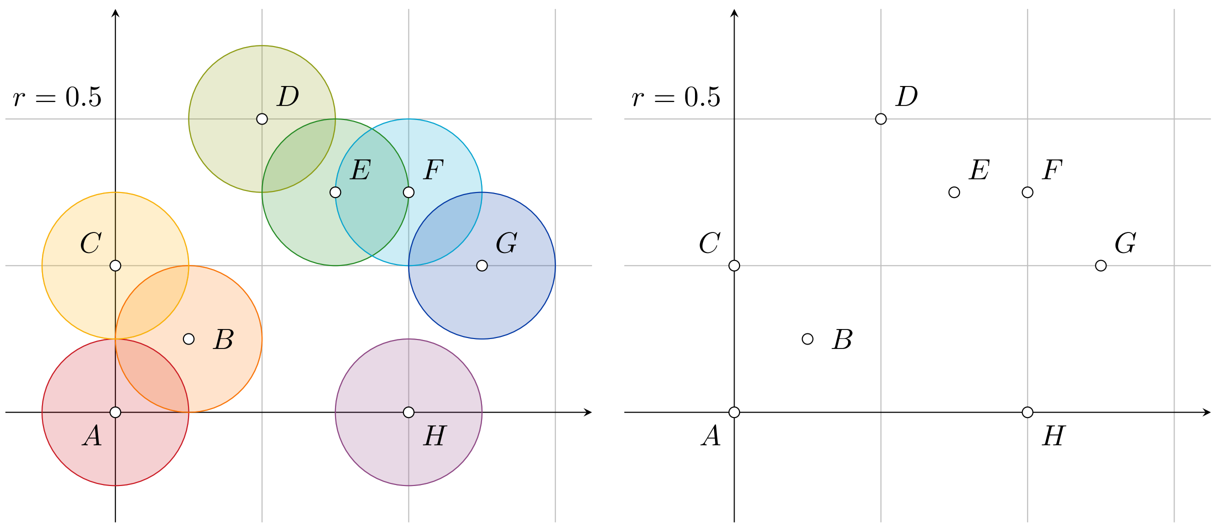
2. Let $S = \{A(0,0), B(0.5,0.5), C(0,1), D(1,2), E(1.5,1.5), F(2,1.5), G(2.5,1), H(2,0)\} \subset \mathbb{R}^2$. Build the Čech complex $\text{Cech}(S,r)$ for

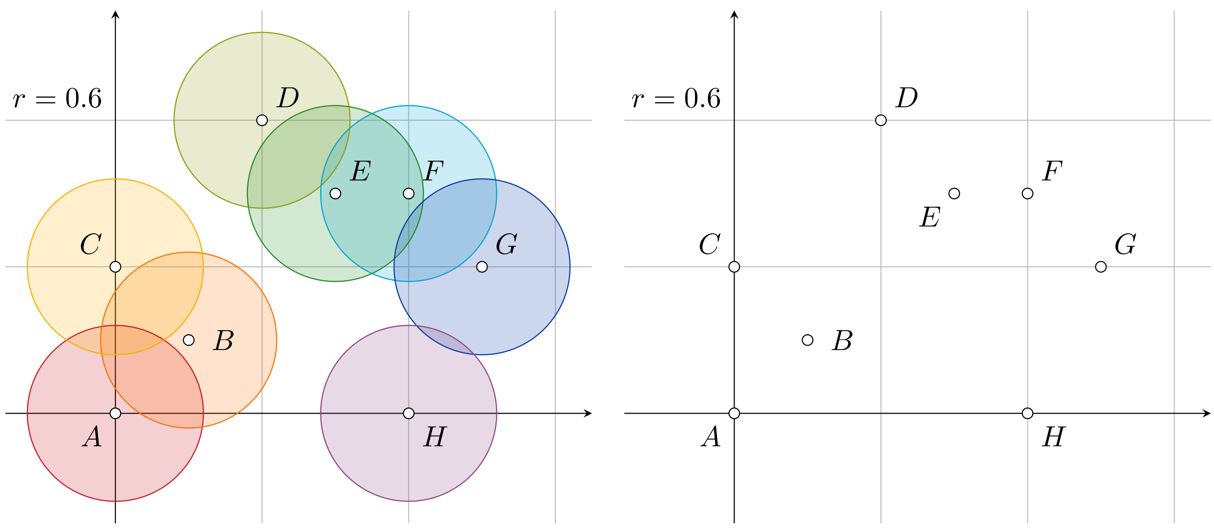
   (a) $r = 0.5$,

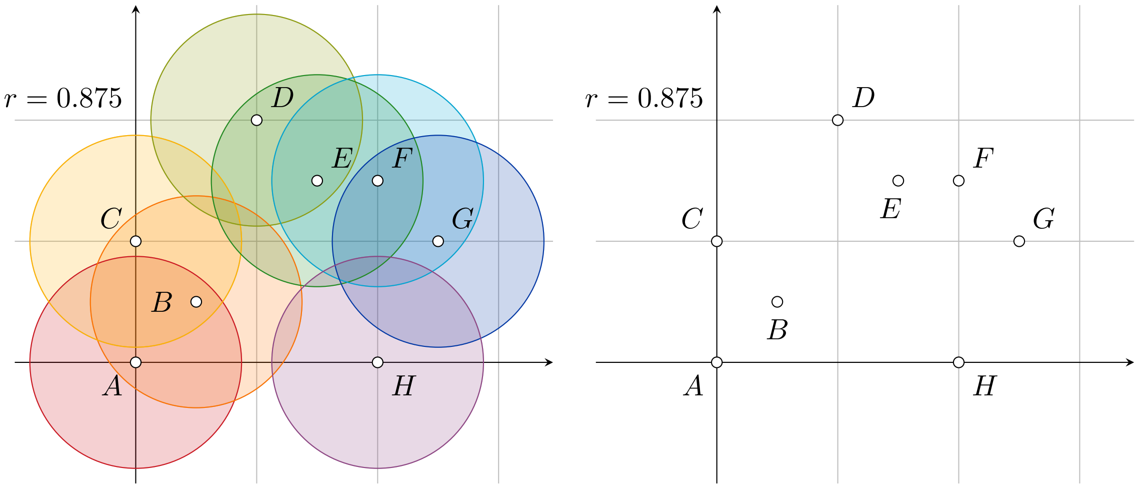
   (b) $r = 0.6$,

   (c) $r = 0.875$.

   In each case list all the simplices and determine its dimension.





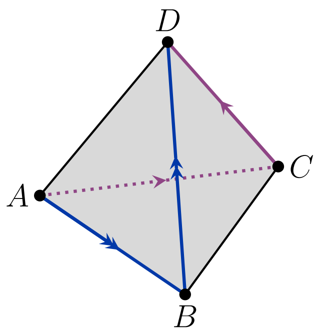


3. The simplicial complexes X and Y are given as lists of simplices:

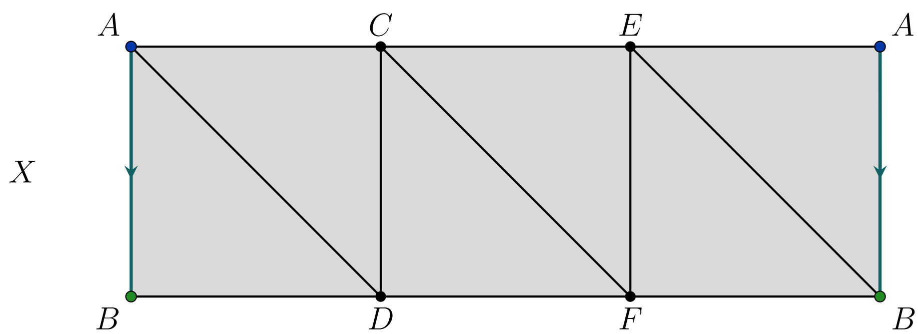
$$\begin{aligned} X &= \{A, B, C, AB, AC, BC\}, \\ Y &= \{A, B, C, D, AB, AD, BC, CD\}. \end{aligned}$$

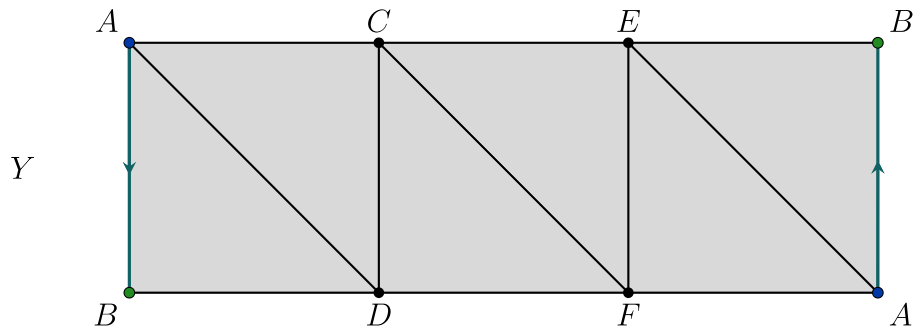
   (a) Construct the cones CX and CY by listing all the simplices.
   (b) Find the sequences of collapses that simplify CX and CY as much as possible.
   (c) Is CX a collapsible complex for all X?

4. Let $X = \Delta^3$ be the standard 3-simplex (tetrahedron) with vertices A, B, C and D. We obtain Y by identifying the edges AB and BD and the edges AC and CD (preserving the ordering of vertices). Show that Y collapses onto a Klein bottle.



5. Given the following triangulations of the cylinder X and the Moebius band Y, find a sequence of elementary collapses that simplifies them as much as possible, then compute the homology groups $H_*(X)$ and $H_*(Y)$.





6. For the simplicial complex X in the figure below

(a) write down the chain groups $\mathcal{C}_n$,

(b) determine the boundary homomorphisms $\partial_n \colon \mathcal{C}_n \to \mathcal{C}_{n-1}$,

(c) find the cycles $Z_n = \ker \partial_n$,

(d) find the boundaries $B_n = \operatorname{im} \partial_n$,

(e) determine the simplicial homology groups with $\mathbb{Z}$ coefficients, $H_n(X;\mathbb{Z})$,

(f) determine the simplicial homology groups with $\mathbb{Z}_2$ coefficients, $H_n(X;\mathbb{Z}_2)$,

(g) determine the Betti numbers of X and

(h) compute the Euler characteristic of X.