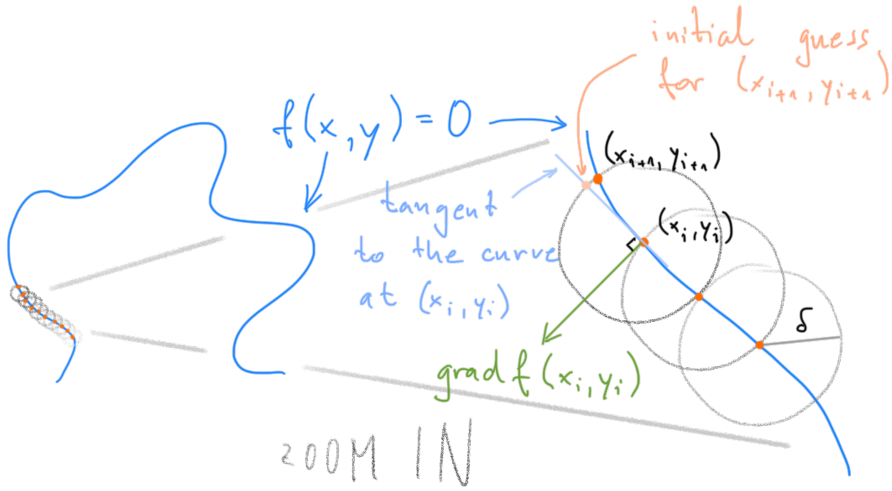
2.



We are trying to construct orange dots. Suppose we are currently at the point (x_i, y_i). To construct the next orange dot, we'll move along the tangent through the point (x_i, y_i) for a step of length δ. This will give us an initial guess, and Newton's iteration will (hopefully) converge to the next orange dot - the point (x_{i+1}, y_{i+1}).

Let's denote $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} := (\text{grad} f)(x_i, y_i)$. Then the vector $\frac{\delta}{\|\vec{n}\|} \begin{bmatrix} -n_2 \\ n_1 \end{bmatrix}$ is tangent to our curve and of length δ. We'll plug

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} + \frac{\delta}{\|\vec{n}\|} \begin{bmatrix} -n_2 \\ n_1 \end{bmatrix}$$

(as our new initial guess) into Newton's iteration to get (x_{i+1}, y_{i+1}).

We still need the function $\vec{F}$ (and it's Jacobi mat.) to run Newton's method. The orange dot (x_{i+1}, y_{i+1}) is a solution of the system of equations:

$$\text{system I.} \begin{cases} f(x,y) = 0, \\ (x-x_i)^2 + (y-y_i)^2 - \delta^2 = 0, \end{cases}$$

ie. the function $\vec{F}$ is

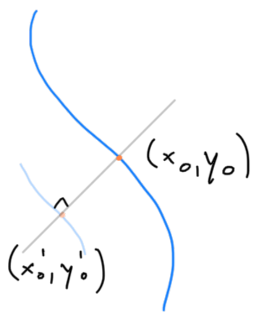
$$\vec{F}(\vec{x}) = \vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x,y) \\ (x-x_i)^2 + (y-y_i)^2 - \delta^2 \end{bmatrix}.$$

Its Jacobi matrix is

$$J\vec{F}(\vec{x}) = J\vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ 2(x-x_i) & 2(y-y_i) \end{bmatrix} = \begin{bmatrix} (\operatorname{grad} f)^T \\ 2(x-x_i) \quad 2(y-y_i) \end{bmatrix}.$$

And that's all the data we need to run the Newton's iteration to find the "next" point on our curve.

A minor problem: The starting point (x_0', y_0') is probably known only approximately (and does not lie on the curve). We'll use (x_0', y_0') as the initial guess for Newton's iteration on the system:



$$\text{system II.} \quad \begin{cases} f(x,y) = 0, \\ \begin{bmatrix} -n_2 \\ n_1 \end{bmatrix} \cdot \begin{bmatrix} x - x_0' \\ y - y_0' \end{bmatrix} = 0, \end{cases}$$

where $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = (\operatorname{grad} f)(x_0', y_0')$.

(Why does this make sense? What is the corresponding Jacobi matrix in this case?)

Note (!): The above is only one of possible solutions. There are other sensible ways of determining the next initial guess (for system I.) and there are other ways of finding the first point on the curve (and having a different system II.). There is (probably) no "best" way of determining the next initial guess – you can always find a curve for which the method will fail.