5. ORTOGONALNOST

ortogonalnost = pravokotnost

Kaj se bomo naučili:
- iskanje baz/množic, v katerih so vsi vektorji paroma pravokotni
- pravokotne projekcije
- ortogonalne matrike/preslikave

5.1 ORTOGONALNOST VEKTORJEV ← potrebujemo skalarni produkt

- v $\mathbb{R}^n$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

"skalarni produkt"

navadno matrično množenje vrstice $\vec{x}^T$ s stolpcem $\vec{y}$

- v $\mathbb{C}^n$: $\vec{x}, \vec{y} \in \mathbb{C}^n$: $\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2} + \dots + x_n \overline{y_n}$ ← konjugirana vrednost

$$\left\langle \begin{bmatrix} 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\rangle = (1+i)(1-i) + 1 \cdot 1 = 1 - i^2 + 1 = 3 \geq 0$$

- v $\mathbb{R}^{m \times n}$: $A, B \in \mathbb{R}^{m \times n}$: $\langle \underline{A}, \underline{B} \rangle = \text{sled}(\underbrace{A^T B}_{n \times n}) = (A^T B)_{11} + (A^T B)_{22} + \dots + (A^T B)_{nn}$

sled ← vsota diagonalnih elementov matrike

Za vsak skalarni produkt $\|x\| = \sqrt{\langle x, x \rangle}$ je dolžina/norma x.

Def: Pravimo, da sta u in v pravokotna/ortogonalna, če $\langle u, v \rangle = 0$. Pisali bomo $u \perp v$.

Pravimo, da je u enotski/normiran, če $\|u\| = 1$.

Def: Množica vektorjev $\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$ je ortogonalna množica, če $\vec{v}_i \perp \vec{v}_j$ za $\forall i \neq j$.

Množica $\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$ je ortonormirana množica, če
- $\vec{v}_i \perp \vec{v}_j$ za $\forall i \neq j$ in
- $\|\vec{v}_i\| = 1$ za $i = 1, \dots, k$

Primer: $\{\vec{i}, \vec{j}, \vec{k}\}$ je ortonormirana množica v $\mathbb{R}^3$.

Odslej živimo v $\mathbb{R}^n$.

Lastnosti:

① Če je $\{\vec{v}_1, \dots, \vec{v}_k\}$ ortogonalna množica neničelnih vektorjev v $\mathbb{R}^n$, potem so $\vec{v}_1, \dots, \vec{v}_k$ linearno neodvisni.

dokaz: $\vec{v}_i^T \cdot \backslash \quad \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_k \vec{v}_k = \vec{0}$

$i \in \{1, \dots, k\} \quad \alpha_1 \vec{v}_i^T \vec{v}_1 + \alpha_2 \vec{v}_i^T \vec{v}_2 + \dots + \alpha_k \vec{v}_i^T \vec{v}_k = 0$

Ker $\vec{v}_i \perp \vec{v}_j$ za $i \neq j$, je $\vec{v}_i^T \vec{v}_j = 0$ za vse $j \neq i$.

$$\alpha_i \underbrace{\vec{v}_i^T \vec{v}_i}_{\|\vec{v}_i\|^2 \neq 0} = 0 \quad \Big/ : \vec{v}_i^T \cdot \vec{v}_i$$

$$\alpha_i = 0$$

To naredimo za vsak $i = 1, \dots, k$, iz česar sledi $\alpha_1 = \dots = \alpha_k = 0$.

$\Rightarrow \vec{v}_1, \dots, \vec{v}_k$ so lin. neodvisni.

Def: Množica vektorjev $\{\vec{b}_1, \dots, \vec{b}_n\}$ je ortonormirana baza $\mathbb{R}^n$ (ONB), če

- $\vec{b}_i \perp \vec{b}_j$ za vsaka $i \neq j$,
- $\|\vec{b}_i\| = 1$ za $i = 1, \dots, n$
- ($\{\vec{b}_1, \dots, \vec{b}_n\}$ je baza $\mathbb{R}^n$)

ortogon. $\Rightarrow$ lin. neodv.
n lin. neodv. $\Rightarrow$ baza v $\mathbb{R}^n$

② Če $\{\vec{b}_1, \dots, \vec{b}_n\}$ ONB $\mathbb{R}^n$ $\Rightarrow$ vsak $\vec{v} \in \mathbb{R}^n$ lahko zapišemo (na en sam način) kot

$$\vec{b}_i^T \backslash \quad \vec{v} = \alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \dots + \alpha_n \vec{b}_n$$

$$\vec{b}_i^T \vec{v} = \alpha_i \underbrace{\vec{b}_i^T \vec{b}_i}_{=1}$$

(kot prej) $\leftarrow \vec{b}_i \perp \vec{b}_j \ \forall j \neq i$

$\leftarrow \|\vec{b}_i\| = 1 \ \forall i$

$$\boxed{\alpha_i = \vec{b}_i^T \vec{v}} = \vec{v}^T \vec{b}_i$$

$\leftarrow$ KUL. Ni nam več treba reševati lin. sistemov.

$$\boxed{\vec{v} = (\vec{v}^T \vec{b}_1)\vec{b}_1 + (\vec{v}^T \vec{b}_2)\vec{b}_2 + \dots + (\vec{v}^T \vec{b}_n)\vec{b}_n}$$

Primer: a) Ali je
$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}_{\vec{v}_3}, \underbrace{\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}}_{\vec{v}_4} \right\}$$
ONB za $\mathbb{R}^4$?

ne, saj je $\left\| \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \right\|^2 = 1^2 + 0^2 + 1^2 + 0^2 = 2 \Rightarrow \left\| \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \right\| = \sqrt{2}$.

b) Ali je $\mathcal{B}$ je ortogonalna množica v $\mathbb{R}^4$?

$\vec{v}_1^T \vec{v}_2 = 0, \ \vec{v}_1^T \vec{v}_3 = 0, \ \dots, \ \vec{v}_3^T \vec{v}_4 = 0 \Rightarrow$ DA.

c) $\{\vec{v}_1, \dots, \vec{v}_4\}$ ortogonalna množica $\Rightarrow \vec{v}_1, \dots, \vec{v}_4$ lin. neodvisni
$\Rightarrow \mathcal{B}$ je baza $\mathbb{R}^4$

Ker $\|\vec{v}_i\| = \sqrt{2}$ za $i = 1,2,3,4$, je
$$\mathcal{B}' = \left\{ \underbrace{\tfrac{1}{\sqrt{2}}\vec{v}_1}_{\vec{u}_1}, \underbrace{\tfrac{1}{\sqrt{2}}\vec{v}_2}_{\vec{u}_2}, \underbrace{\tfrac{1}{\sqrt{2}}\vec{v}_3}_{\vec{u}_3}, \underbrace{\tfrac{1}{\sqrt{2}}\vec{v}_4}_{\vec{u}_4} \right\}$$
je ONB $\mathbb{R}^4$.

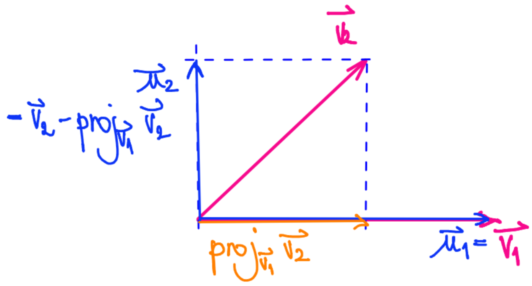
d) Zapišimo $\vec{w} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ kot lin. komb. vektorjev $\underbrace{\vec{u}_1, \vec{u}_2, \vec{u}_3 \text{ in } \vec{u}_4}_{\text{ONB}}$.

$$\vec{w}^T\vec{u}_1 = \frac{1}{\sqrt{2}}\,[1\ 2\ 3\ 4]\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}(1+3) = \frac{4}{\sqrt{2}} \qquad \vec{w}^T\vec{u}_3 = \frac{1}{\sqrt{2}}(2+4) = \frac{6}{\sqrt{2}}$$

$$\vec{w}^T\vec{u}_2 = \frac{1}{\sqrt{2}}\,[1\ 2\ 3\ 4]\cdot\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}(1-3) = -\frac{2}{\sqrt{2}} \qquad \vec{w}^T\vec{u}_4 = \frac{1}{\sqrt{2}}(2-4) = -\frac{2}{\sqrt{2}}$$

$$\Rightarrow \quad \vec{w} = \frac{4}{\sqrt{2}}\vec{u}_1 - \frac{2}{\sqrt{2}}\vec{u}_2 + \frac{6}{\sqrt{2}}\vec{u}_3 - \frac{2}{\sqrt{2}}\vec{u}_4 = \sqrt{2}\left(2\vec{u}_1 - \vec{u}_2 + 3\vec{u}_3 - \vec{u}_4\right)$$

Kako poiščemo ortogonalno množico?



$\{\vec{v}_1, \vec{v}_2\}$ lin. neodv.

$\Downarrow$

$\{\vec{u}_1, \vec{v}_2 - \text{proj}_{\vec{u}_1}\vec{v}_2\}$ ortogonalna in razpenjata isto ravnino kot vektorja $\vec{v}_1$ in $\vec{v}_2$

Gram-Schmidtov postopek

vhodni podatki: lin. neodvisni vektorji $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$

izhodni podatki: ortogonalna množica vektorjev $\vec{u}_1, \dots, \vec{u}_k \in \mathbb{R}^n$, za katere velja $\mathcal{L}\{\vec{u}_1, \dots, \vec{u}_j\} = \mathcal{L}\{\vec{v}_1, \dots, \vec{v}_j\}$ za $j = 1, \dots, k$

$$\vec{u}_1 = \vec{v}_1$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}\vec{v}_2 = \vec{v}_2 - \frac{\vec{v}_2^T\vec{u}_1}{\vec{u}_1^T\vec{u}_1}\vec{u}_1$$

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1}\vec{v}_3 - \text{proj}_{\vec{u}_2}\vec{v}_3 = \vec{v}_3 - \frac{\vec{v}_3^T\vec{u}_1}{\vec{u}_1^T\vec{u}_1}\vec{u}_1 - \frac{\vec{v}_3\vec{u}_2}{\vec{u}_2^T\vec{u}_2}\vec{u}_2$$

$$\vdots$$

$$\vec{u}_k = \vec{v}_k - \text{proj}_{\vec{u}_1}\vec{v}_k - \text{proj}_{\vec{u}_2}\vec{v}_k - \dots - \text{proj}_{\vec{u}_{k-1}}\vec{v}_k = \vec{v}_k - \sum_{i=1}^{k-1}\frac{\vec{v}_k^T\vec{u}_i}{\vec{u}_i^T\vec{u}_i}\vec{u}_i$$

(t.j. vsakemu nasl. vektorju odštejemo projekcije na vse dosedanje nove vektorje.)

Primer: Naj bo $U = \mathcal{L}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq \mathbb{R}^4$, kjer $\vec{v}_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$. Poiščimo kakšno ONB prostora U.

1. način $\quad \vec{u}_1 = \vec{v}_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2^T \vec{u}_1}{\vec{u}_1^T \vec{u}_1}\vec{u}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} \qquad (\text{res } \vec{u}_2 \perp \vec{u}_1)$$

$$\vec{u}_3 = \vec{v}_3 - \frac{\vec{v}_3^T \vec{u}_1}{\vec{u}_1^T \vec{u}_1}\vec{u}_1 - \frac{\vec{v}_3^T \vec{u}_2}{\vec{u}_2^T \vec{u}_2}\vec{u}_2 = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} - \frac{0}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{1}{\frac{1}{2}}\begin{bmatrix}1/2\\0\\-1/2\\0\end{bmatrix} =$$

$$= \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \qquad (\text{res } \vec{u}_3 \perp \vec{u}_1 \text{ in } \vec{u}_3 \perp \vec{u}_2)$$

($\vec{v}_1, \vec{v}_2, \vec{v}_3$ res lin. neodv $\Rightarrow$ $\dim U = 3$)

$\Rightarrow$ $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ baza U, ortogonalna množica

$$\left.\begin{aligned}
\vec{w}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\\
\vec{w}_2 &= \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\\
\vec{w}_3 &= \frac{\vec{u}_3}{\|\vec{u}_3\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}
\end{aligned}\right\} \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} \text{ ONB prostora } U.$$

2. način: $\vec{v}_1{}' = \vec{v}_2$
$\vec{v}_2{}' = \vec{v}_1$
$\vec{v}_3{}' = \vec{v}_3$ (premešali vrstni red)

$$\vec{u}_1' = \vec{v}_1' = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$\vec{u}_2' = \vec{v}_2' - \mathrm{proj}_{\vec{u}_1'}\vec{v}_2' = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \qquad (\vec{u}_2' \perp \vec{u}_1')$$

$$\vec{u}_3' = \vec{v}_3' - \mathrm{proj}_{\vec{u}_1'}\vec{v}_3' - \mathrm{proj}_{\vec{u}_2'}\vec{v}_3' = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{-1}{1}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \qquad (\vec{u}_3' \perp \vec{u}_1', \vec{u}_3' \perp \vec{u}_2')$$

$\Rightarrow$ Gram-Schmidtov postopek je odvisen od vrstnega reda vhodnih vektorjev.

$$\Rightarrow \left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\} \text{ ONB prostora } U.$$

5.2 ORTOGONALNI KOMPLEMENT

Def: Naj bo U vektorski podprostor v $\mathbb{R}^n$. Potem množico

$$U^\perp = \{\vec{v} \in \mathbb{R}^n;\ \vec{v} \perp \vec{u} \text{ za } \forall \vec{u} \in U\}$$

imenujemo ortogonalni komplement prostora U.

Lastnosti:

1) $U^\perp$ je vektorski podprostor v $\mathbb{R}^n$. (DN)

2) $U \cap U^\perp = \{\vec{0}\}$

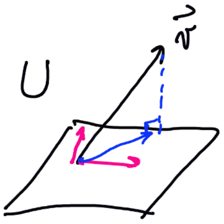
(Če $\vec v \in U^\perp$: $\vec v \perp \vec u$ za vsak $\vec u \in U$
in če $\vec v \in U$: $\vec v \perp \vec v$ (saj $\vec v \in U$)
$\vec v^T \vec v = 0$
$\|\vec v\|^2 = 0$
$\vec v = \vec 0 \Rightarrow \{\vec 0\} = U \cap U^\perp$)

3) Vsak $\vec v \in \mathbb{R}^n$ lahko na en sam način zapišemo kot
$\vec v = \vec u + \vec u'$, kjer $\vec u \in U$, $\vec u' \in U^\perp$ ($\mathbb{R}^n = U \oplus U^\perp$)

(Zakaj? Naj bo $\{\vec b_1, \dots, \vec b_k\}$ ONB U. Definirajmo

$$\vec z = (\vec v^T \vec b_1)\vec b_1 + (\vec v^T \vec b_2)\vec b_2 + \dots + (\vec v^T \vec b_k)\vec b_k \in U$$

$$\vec w = \vec v - \vec z$$



CILJ: $\vec w \in U^\perp$. Pokažimo: $\vec w \perp \vec b_j$:

$$\vec w^T \vec b_j = (\vec v - \vec z)^T \vec b_j =$$

$$= \vec v^T \vec b_j - \vec z^T \vec b_j =$$

$$= \vec v^T \vec b_j - \left((\vec v^T \vec b_1)\underbrace{\vec b_1^T \vec b_j} + (\vec v^T \vec b_2)\underbrace{\vec b_2^T \vec b_j} + \dots + (\vec v^T \vec b_k)\underbrace{\vec b_k^T \vec b_j}\right)$$

vsi 0 razen $\vec b_j^T \vec b_j = 1$

$$= \vec v^T \vec b_j - (\vec v^T \vec b_j)\underbrace{\vec b_j^T \vec b_j}_{=1} = 0$$

$\Rightarrow \vec w \perp \vec b_j$
$\Rightarrow \vec w \in U^\perp$

$$\Rightarrow \vec v = \underset{\in U}{\vec z} + \underset{\in U^\perp}{\vec w}$$

Kaj, če $\vec v = \underset{\in U}{\vec z} + \underset{\in U^\perp}{\vec w} = \underset{\in U}{\vec z'} + \underset{\in U^\perp}{\vec w'}$? Potem $\underbrace{\vec z - \vec z'}_{\in U} = \underbrace{\vec w' - \vec w}_{\in U^\perp} \Rightarrow$

$\Rightarrow \vec z - \vec z' = \vec 0$ in $\vec w' - \vec w = \vec 0$
$\Rightarrow \vec z = \vec z'$ in $\vec w = \vec w'$
$\Rightarrow$ zapis res enoličen.)

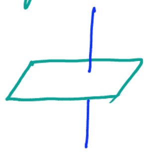
4) $\{\vec b_1, \vec b_2, \dots, \vec b_n\}$ ONB $\mathbb{R}^n$ in $U = \mathcal{L}\{\vec b_1, \dots, \vec b_k\}$ ($\Rightarrow \{\vec b_1, \dots, \vec b_k\}$ ONB U)
$U^\perp = \mathcal{L}\{\vec b_{k+1}, \dots, \vec b_n\}$ ($\Rightarrow \{\vec b_{k+1}, \dots, \vec b_n\}$ ONB $U^\perp$)

Za $\vec v \in \mathbb{R}^n$ je $\mathrm{proj}_U \vec v = (\vec v^T \vec b_1)\vec b_1 + \dots + (\vec v^T \vec b_k)\vec b_k$ in
$\mathrm{proj}_{U^\perp} \vec v = (\vec v^T \vec b_{k+1})\vec b_{k+1} + \dots + (\vec v^T \vec b_n)\vec b_n$.

5) $\dim U^\perp = n - \dim U$

6) $(U^\perp)^\perp = U$

Primera: ① Naj $\Sigma: ax+by+cz=0$ v $\mathbb{R}^3$, potem je $\Sigma^\perp = \mathcal{L}\left\{\begin{bmatrix}a\\b\\c\end{bmatrix}\right\}$.



② $A \in \mathbb{R}^{m\times n}$. Kaj je $C(A)^\perp$?

$A = [A^{(1)}, \dots, A^{(n)}]$ ← stolpci matrike A

Izberimo $\vec{x} \in C(A)^\perp \iff \vec{x} \perp A^{(i)}, \ i=1,\dots,n$

$$\iff (A^{(i)})^T \vec{x} = 0, \quad i=1,\dots,n$$

$$\iff \underbrace{\begin{bmatrix} A^{(1)T} \\ A^{(2)T} \\ \vdots \\ A^{(n)T} \end{bmatrix}}_{A^T} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

$$\iff A^T \vec{x} = \vec{0}$$

$$\iff \vec{x} \in N(A^T)$$

$$\boxed{C(A)^\perp = N(A^T)} \qquad \boxed{C(A^T)^\perp = N(A)}$$

Spomnimo se od zadnjič:

U vekt. podpr. v $\mathbb{R}^n$ in $\{\vec{b}_1, \dots, \vec{b}_k\}$ ONB za U

$$\vec{b}_i^T \vec{b}_j = \begin{cases} 1, & \text{če } i=j, \\ 0, & \text{če } i \neq j. \end{cases}$$

Za vsak $\vec{v} \in \mathbb{R}^n$:

$$\text{proj}_U \vec{v} = (\vec{v}^T\vec{b}_1)\vec{b}_1 + (\vec{v}^T\vec{b}_2)\vec{b}_2 + \dots + (\vec{v}^T\vec{b}_k)\vec{b}_k =$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k \\ | & | & & | \end{bmatrix}}_{Q \in \mathbb{R}^{n\times k}} \begin{bmatrix} \vec{v}^T\vec{b}_1 \\ \vec{v}^T\vec{b}_2 \\ \vdots \\ \vec{v}^T\vec{b}_k \end{bmatrix} =$$

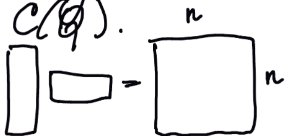
stolpci $\vec{b}_i \in \mathbb{R}^n$, $\vec{v}^T\vec{b}_i \in \mathbb{R}$, $\begin{bmatrix} \vec{v}^T\vec{b}_1 \\ \vdots \\ \vec{v}^T\vec{b}_k \end{bmatrix} \in \mathbb{R}^k$

$$\vec{v}^T\vec{b}_1 = \vec{b}_1^T\vec{v}$$
$$\vec{v}^T\vec{b}_2 = \vec{b}_2^T\vec{v}$$
$$\vdots$$
$$\vec{v}^T\vec{b}_k = \vec{b}_k^T\vec{v}$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k \\ | & | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \vec{b}_1^T \\ \vec{b}_2^T \\ \vdots \\ \vec{b}_k^T \end{bmatrix}}_{Q^T} \vec{v} = (QQ^T)\vec{v} \Rightarrow \text{proj}_U \vec{v} = \boxed{(QQ^T)}\vec{v}$$

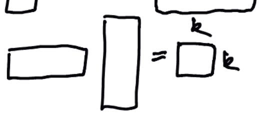
matrika projekcije na $U = \mathcal{L}\{\vec{b}_1, \dots, \vec{b}_k\} = C(Q)$

$\Rightarrow$ Matrika QQ^T je matrika projekcije ($\mathbb{R}^n \to \mathbb{R}^n$) na $C(Q)$, če ima $Q \in \mathbb{R}^{n\times k}$ ($k \le n$) stolpce, ki tvorijo ONB za $C(Q)$.

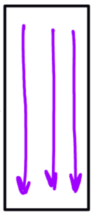
$QQ^T =$ 

$Q =$ 

Kaj je Q^TQ?

$Q^TQ =$ 

$$Q^TQ = \begin{bmatrix} \vec{b}_1^T \\ \vec{b}_2^T \\ \vdots \\ \vec{b}_k^T \end{bmatrix} [\vec{b}_1\ \vec{b}_2 \dots \vec{b}_k] = \begin{bmatrix} \vec{b}_1^T\vec{b}_1 & \vec{b}_1^T\vec{b}_2 & \cdots & \vec{b}_1^T\vec{b}_k \\ \vec{b}_2^T\vec{b}_1 & \vec{b}_2^T\vec{b}_2 & \cdots & \vec{b}_2^T\vec{b}_k \\ \vdots & \vdots & \ddots & \vdots \\ \vec{b}_k^T\vec{b}_1 & \vec{b}_k^T\vec{b}_2 & \cdots & \vec{b}_k^T\vec{b}_k \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} = I_k$$

Torej: $Q=$  stolpci Q so ONB za $C(Q)$ $\Rightarrow$

- $QQ^T \in \mathbb{R}^{n\times n}$ ... matrika projekcije na $C(Q)$
- $I_k = Q^TQ \in \mathbb{R}^{k\times k}$

Primer: Zapišimo matriko, ki ustreza pravokotni projekciji iz $\mathbb{R}^3$ na ravnino $\Sigma: x+y+2z=0$.

1. korak: poiščimo ONB Σ (ali uganemo ali pa naredimo GS)

izberemo $\vec{a} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \in \Sigma$ in naredimo GS:

$$\vec{u}_1 = \vec{a} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \vec{b} - \mathrm{proj}_{\vec{u}_1}\vec{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{u}_1 \perp \vec{u}_2$$

$\{\vec{w}_1, \vec{w}_2\}$ ONB Σ, $\vec{w}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

2. korak $Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix}$ $\qquad C(Q) = \Sigma$

3. korak $P = QQ^T = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$

↳ projekcija na Σ

- kam se s projekcijo slika $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$? $\quad P\vec{c} = P\cdot\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{6} \\ \frac{3}{6} \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Q kvadratna

Definicija: Matriki $Q \in \mathbb{R}^{n\times n}$ pravimo ortogonalna matrika, če velja $QQ^T = I_n = Q^TQ$.

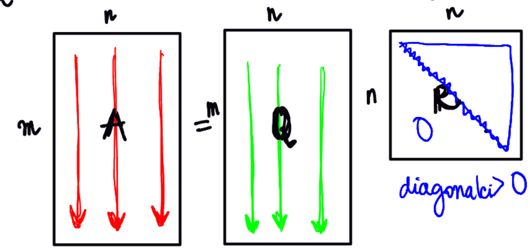
velja: $\Leftrightarrow$ Q ima stolpce, ki tvorijo ONB za $\mathbb{R}^n$ ($Q^{(i)} \perp Q^{(j)}$ za $i \neq j$, $\|Q^{(i)}\| = 1$)

$\Leftrightarrow Q^{-1} = Q^T$

- $\det Q = \pm 1$

5.3 QR razcep matrike (faktorizacija $A=QR$)

Izrek (QR razcep)
Naj bo $A \in \mathbb{R}^{m\times n}$, $m \geq n$, $rk(A)=n$ (t.j. stolpci A so lin. neodv, oz. $\dim C(A)=n$)
Potem obstajata matriki
- $Q\in\mathbb{R}^{m\times n}$ matrika, katere stolpci tvorijo ONB $C(Q)=C(A)$
- $R\in\mathbb{R}^{n\times n}$ zgornje trikotna matrika, katere diagonalni elementi so pozitivni



$C(Q)=C(A)$
stolpci Q so ONB za $C(Q)$

Zakaj to res? Vse že vemo

$A=[A^{(1)},\dots,A^{(n)}]$ in na $\{A^{(1)},\dots,A^{(n)}\}$ naredimo GS + normiramo, dobimo $\vec{q}_1,\dots,\vec{q}_n$, da $\vec{q}_i \perp \vec{q}_j$ za $i\neq j$
$\|\vec{q}_i\|=1$
$\mathcal{L}\{\vec{q}_1,\dots,\vec{q}_n\}=C(A)$

$Q=[\vec{q}_1,\dots,\vec{q}_n]$ ima vse željene lastnosti

$$\vec{u}_1=\vec{a}_1$$
$$\vec{u}_2=\vec{a}_2-\mathrm{proj}_{\vec{u}_1}\vec{a}_2=\vec{a}_2-\alpha_1\vec{u}_1$$
$$\vdots$$
$$\vec{u}_j=\vec{a}_j-\alpha_1\vec{u}_1-\alpha_2\vec{u}_2-\dots-\alpha_{j-1}\vec{u}_{j-1} \quad \Big/ \text{normiramo } \frac{1}{\|\vec{u}_j\|}$$
$$\vec{q}_j=\frac{1}{\|\vec{u}_j\|}\vec{a}_j-\beta_1\vec{q}_1-\beta_2\vec{q}_2-\dots-\beta_{j-1}\vec{q}_{j-1}$$
$$\vec{a}_j=\gamma_1\vec{q}_1+\dots+\gamma_{j-1}\vec{q}_{j-1}+\underbrace{\|\vec{u}_j\|}_{=\gamma_j}\vec{q}_j+0\cdot\vec{q}_{j+1}+\dots+0\cdot\vec{q}_n$$

$\to$ R je sestavljena iz $\gamma_1,\dots,\gamma_j$ v j-tem stolpcu

2\. način: Če $Q^T \backslash \; A=QR$ (po G-S)

Q ima ON stolpce $\Rightarrow Q^TQ=I_n$

$$Q^TA=(\underbrace{Q^TQ}_{=I_n})R=R \Rightarrow \boxed{R=Q^TA}$$

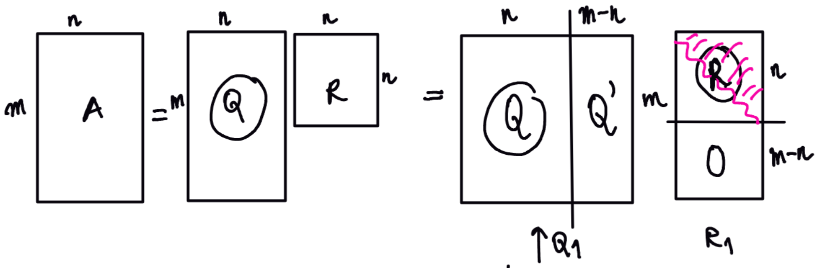
Primer: Poiščimo QR razcep matrike $A=\begin{bmatrix}-1&-2\\1&0\\0&1\end{bmatrix}\in\mathbb{R}^{3\times 2}$.

Ali sta stolpca A lin. neodvisna? DA, očitno.

$Q=\begin{bmatrix}-1/\sqrt{2}&-1/\sqrt{3}\\1/\sqrt{2}&-1/\sqrt{3}\\0&1/\sqrt{3}\end{bmatrix}$, že vemo iz prejšnjega primera.

$$R=Q^TA=\begin{bmatrix}-1/\sqrt{2}&1/\sqrt{2}&0\\-1/\sqrt{3}&-1/\sqrt{3}&1/\sqrt{3}\end{bmatrix}\begin{bmatrix}-1&-2\\1&0\\0&1\end{bmatrix}=\begin{bmatrix}\sqrt{2}&\sqrt{2}\\0&\sqrt{3}\end{bmatrix}\qquad A=QR$$

Variante QR:

- 

- Q ... GS ali Householderjeva zrcaljenja ali Givensove rotacije

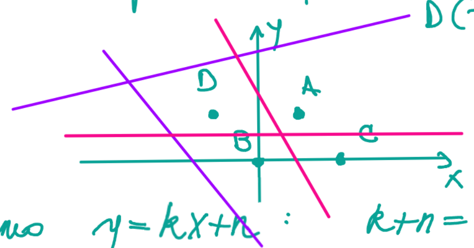
5.4. PREDOLOČENI SISTEMI

$A\in\mathbb{R}^{m\times n}$
$\vec{b}\in\mathbb{R}^{m}$

$\boxed{m>n}$ ← predoločeni sistem

$m<n$

Določimo premico skozi $A(1,1)$, $B(0,0)$, $C(2,0)$, $D(-1,1)$



(Jasno, takšna premica ne obstaja.)

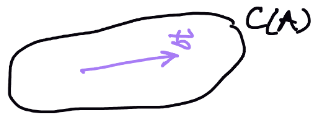
iščemo $y=kx+n$:

$$\begin{aligned}k+n&=1\\n&=0\\2k+n&=0\\-k+n&=1\end{aligned}\qquad k=1\qquad 2k=0\quad k=0$$

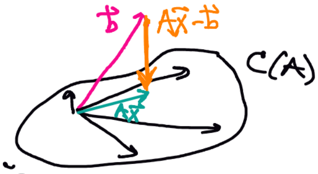
$$\underbrace{\begin{bmatrix}1&1\\0&1\\2&1\\-1&1\end{bmatrix}}_{4\times 2}\begin{bmatrix}k\\n\end{bmatrix}=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$A\vec{x}=\vec{b}$, $A\in\mathbb{R}^{m\times n}$ $m\geq n$

1) Kdaj $A\vec{x}=\vec{b}$ ima rešitev? $\vec{b}\in C(A)$



2) Če $\vec{b}\notin C(A)$, potem ne obstaja tak $\vec{x}$, da bi $\vec{b}=A\vec{x}$.



Zato iščemo tak $\vec{x}$, da "$\vec{b}\approx A\vec{x}$"

$\|A\vec{x}-\vec{b}\|$ čim manjša.

Takšnemu $\vec{x}_0$, da je $\|A\vec{x}_0-\vec{b}\|\leq\|A\vec{x}-\vec{b}\|$ za vsak $\vec{x}\in\mathbb{R}^n$, pravimo, da je dobljen po metodi najmanjših kvadratov.

T.j. iščemo tak $\vec{x}$, da bo $A\vec{x}=\text{proj}_{C(A)}\vec{b}$

$\vec{e}=A\vec{x}-\vec{b}$, iščemo $\min\|\vec{e}\|=\min\|A\vec{x}-\vec{b}\|$

$\vec{e}\perp C(A)$

$\vec{e}\in C(A)^{\perp}=N(A^T)$ $\Rightarrow$ $A^T\vec{e}=\vec{0}$

$A^T(A\vec{x}-\vec{b})=\vec{0}$

$\boxed{A^TA\vec{x}=A^T\vec{b}}$

$\vec{x}$, ki "najbolje reši" $A\vec{x}=\vec{b}$ je tisti $\vec{x}$, ki je rešitev

$\boxed{A^TA\vec{x}=A^T\vec{b}}$ ← normalni sistem.

Sklep: $\vec{x}$, ki po metodi najmanjših kvadratov da rešitev predoločenega sistema $A\vec{x}=\vec{b}$, je rešitev normalnega sistema $A^TA\vec{x}=A^T\vec{b}$

Določimo premico, ki se najbolje prilega $A(1,1)$, $B(0,0)$, $C(2,0)$, $D(-1,1)$.

$$M=\begin{bmatrix}1&1\\0&1\\2&1\\-1&1\end{bmatrix},\quad \vec{b}=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$M\vec{x}=\vec{b}$ nima rešitev

Zato iščemo rešitve $M^TM\vec{x}=M^T\vec{b}$.

$$M^TM=\begin{bmatrix}1&0&2&-1\\1&1&1&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\\2&1\\-1&1\end{bmatrix}=\begin{bmatrix}6&2\\2&4\end{bmatrix}$$

← M^TM simetrična matrika

$n\times n$ ("majhna")

$$M^T\vec{b}=\begin{bmatrix}0\\2\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c}6&2&0\\2&4&2\end{array}\right]\to\left[\begin{array}{cc|c}1&2&1\\3&1&0\end{array}\right]\to\begin{array}{c}\begin{array}{cc} k & n\end{array}\\ \left[\begin{array}{cc|c}1&2&1\\0&-5&-3\end{array}\right]\end{array}$$

$$M^TM\begin{bmatrix}k\\n\end{bmatrix}=M^T\vec{b}$$

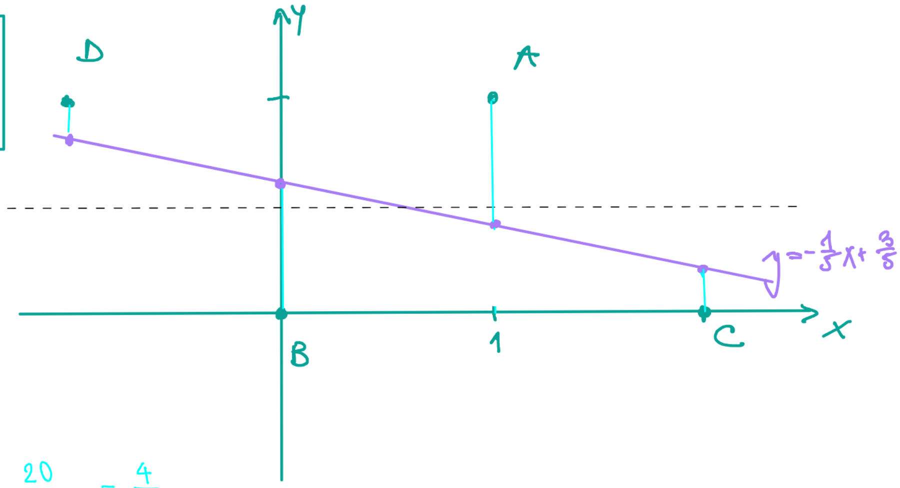
$-5n=-3$

$n=\frac{3}{5}$

$k+2n=1$

$k=1-\frac{6}{5}=-\frac{1}{5}$

$$\Rightarrow \boxed{y = -\frac{1}{5}x + \frac{3}{5}}$$



$$err = \left(\frac{1}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{1}{5}\right)^2 = \frac{20}{25} = \frac{4}{5}$$

(kakšno napako bi naredili s premico $y = \frac{1}{2}$? $err' = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1 > \frac{4}{5}$)

Če $A \in \mathbb{R}^{m \times n}$ polnega ranga ($m \geq n$) (rang $A = n$) (in v praksi je), potem trdimo, da A^TA obrnljiva.

Zakaj to dobro?

$$(A^TA)^{-1} \backslash \quad A^TA\vec{x} = A^T\vec{b}$$

$$\underbrace{(A^TA)^{-1}(A^TA)}_{= I}\vec{x} = (A^TA)^{-1}A^T\vec{b}$$

$$\boxed{\vec{x} = (A^TA)^{-1}A^T\vec{b}}$$ rešitev normalnega sistema

Zakaj A^TA obrnljiva? $A^TA \in \mathbb{R}^{n \times n}$

Če $A^TA\vec{x} = \vec{0}$ za nek $\vec{x} \in \mathbb{R}^n$

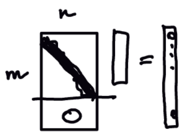
$$\vec{x}^T \cdot \backslash \quad (\vec{x}^TA^T)(A\vec{x}) = 0$$

$$(A\vec{x})^T(A\vec{x}) = 0$$

$$\|A\vec{x}\|^2 = 0$$

$$\|A\vec{x}\| = 0$$

$$A\vec{x} = \vec{0}$$



$\vec{x} = \vec{0}$ je edina rešitev $A\vec{x} = \vec{0}$

$\Rightarrow A^TA$ obrnljiva