Doslej: Če $A \in \mathbb{R}^{n \times n}$ diagonalizabilna $\Rightarrow A = PDP^{-1}$

Če $B \in \mathbb{R}^{n \times n}$ simetrična $\Rightarrow B = QDQ^{-1}$

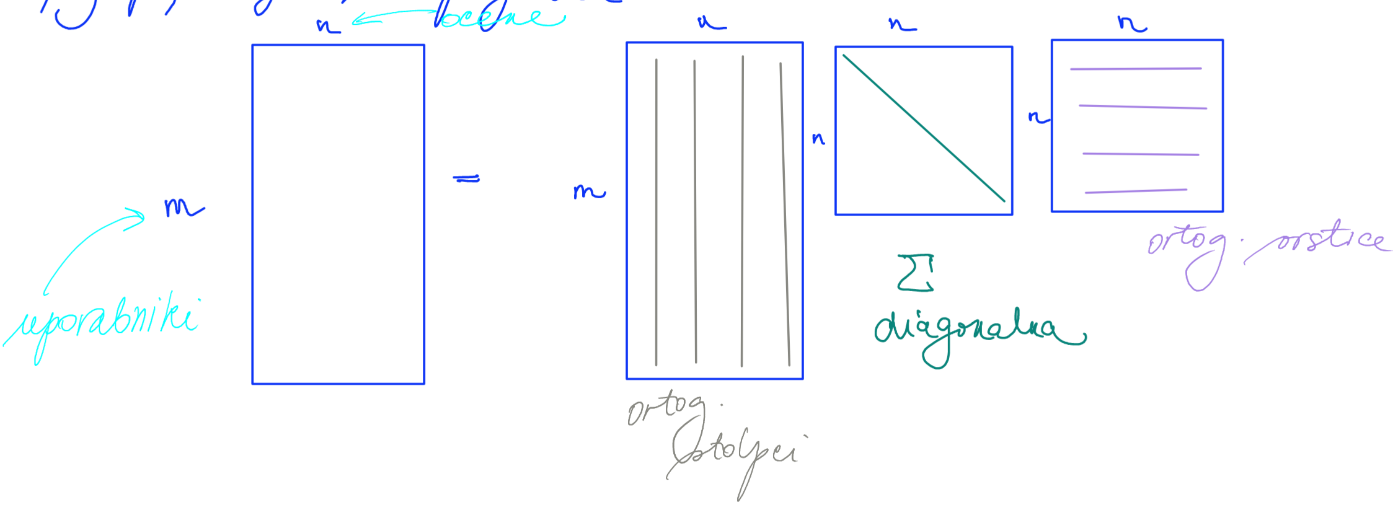
kjer ima Q ~~ortog.~~ ortonormirane stolpce

$\underbrace{[\;]}_{Q^T}\underbrace{[\;]}_{Q} = I \Rightarrow QQ^T = I$

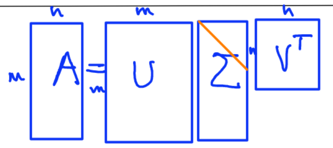
$\Rightarrow Q^{-1} = Q^T$

$\Rightarrow B = QDQ^T$

Kaj pa, če je A poljubna $m \times n$ matrika?



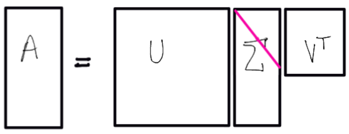
dosle:

	RAZCEP	
$A\in\mathbb{R}^{n\times n}$ simetrična	$A = QDQ^T$	D diagonalna (l. vrednosti) Q ortogonalna (l. vektorji)
$A\in\mathbb{R}^{n\times n}$ diagonalizabilna	$A = PDP^{-1}$	D diagonalna (l. vrednosti) P obrnljiva (l. vektorji)
$A\in\mathbb{R}^{m\times n}$	 $A = U\Sigma V^T$ SVD	Σ diagonalna $m\times n$ matrika U, V ortogonalni matriki

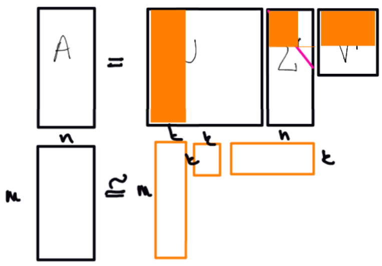
6.6 RAZCEP SINGULARNIH VREDNOSTI (SVD = Singular Value Decomposition)

Radi bi vsako $A\in\mathbb{R}^{m\times n}$ zapisali kot

$\boxed{A = U\Sigma V^T}$, kjer sta $U\in\mathbb{R}^{m\times m}$ in $V\in\mathbb{R}^{n\times n}$ ortogonalni, $\Sigma\in\mathbb{R}^{m\times n}$ diagonalna.



Uporaba:



Tehnikalije: (1) A^TA je simetrična matrika in njene lastne vrednosti so nenegativne

($(A^TA)^T = A^T(A^T)^T = A^TA \Rightarrow A^TA$ simetrična

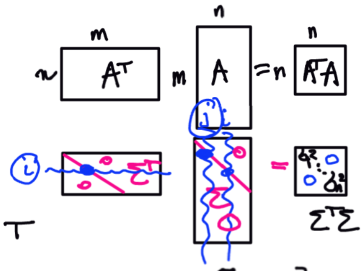
Naj λ lastna vrednost A^TA: $(A^TA)\vec{x} = \lambda\vec{x}$, $\vec{x}\neq\vec{0}$

$$0 \le \|A\vec{x}\|^2 = (A\vec{x})^T(A\vec{x}) = \vec{x}^T A^TA\vec{x} = \vec{x}^T(\lambda\vec{x}) = \lambda\vec{x}^T\vec{x} = \lambda\|\vec{x}\|^2 \quad /:\|\vec{x}\|^2\neq 0$$

$$\lambda = \frac{\|A\vec{x}\|^2 \ge 0}{\|\vec{x}\|^2 \ge 0} \ge 0$$)

Def: Simetrično matriko z nenegativnimi lastnimi vrednostmi imenujemo pozitivno semidefinitna matrika.

(2) Če $A = U\Sigma V^T$ ($A \in \mathbb{R}^{m\times n}$, $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ ortogonalni, $\Sigma \in \mathbb{R}^{m\times n}$ diagonalna),



potem
$$\underset{\substack{\uparrow\\ \text{simetrična}\\ n\times n}}{A^TA} = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T \underbrace{U^TU}_{= I}\Sigma V^T = \underset{\substack{\uparrow\\ n\times n\\ \text{ortogonalna}}}{V}\underset{\substack{\uparrow\\ n\times n\\ \text{diagonalna}}}{(\Sigma^T\Sigma)}V^T$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_n \\ & 0 & \end{bmatrix}$$

lastni vektorji matrike A^TA

$$\boxed{A^TA = V(\Sigma^T\Sigma)V^T}$$

$\Sigma^T\Sigma$ diagonalna

$\Rightarrow$ lastne vrednosti A^TA zapisane na diagonali $\Sigma^T\Sigma = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2\end{bmatrix}$: $\sigma_1^2, \dots, \sigma_n^2$

$\Rightarrow$ diagonalni elementi $\sigma_1, \dots, \sigma_n$ matrike Σ so takšni, da $\sigma_1^2, \dots, \sigma_n^2$ lastne vrednosti A^TA.

(izberemo $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, ..., $\sigma_n = \sqrt{\lambda_n}$, kjer $\lambda_1, \dots, \lambda_n$ lastne vrednosti matrike A^TA)

Velja tudi $AA^T = U\Sigma V^T(U\Sigma V^T)^T = U\Sigma \overbrace{V^TV}^{I}\Sigma^TU^T = U(\Sigma\Sigma^T)U^T$

$$\boxed{AA^T = U(\Sigma\Sigma^T)U^T}$$

simetrična $m\times m$ — diagonalna $m\times m$ — lastni vektorji AA^T

$$\Sigma\Sigma^T = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{bmatrix}\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & 0 \\ & & \sigma_n^2 & \\ & 0 & & 0\end{bmatrix} \Rightarrow$$

$\Rightarrow$ lastne vrednosti AA^T zapisane na diagonali $\Sigma\Sigma^T$:
$\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2, \underbrace{0, \dots, 0}_{m-n}$.

VELJA: Lastne vrednosti $AA^T \in \mathbb{R}^{m\times m}$ in $A^TA \in \mathbb{R}^{n\times n}$ so enake (z izjemo $m-n$ ničelnih lastnih vrednosti.)

$U \in \mathbb{R}^{m\times m}$: po stolpcih l.vekt (ONB) matrike $AA^T \in \mathbb{R}^{m\times m}$
$V \in \mathbb{R}^{n\times n}$: po stolpcih l.vekt (ONB) matrike $A^TA \in \mathbb{R}^{n\times n}$
$\Sigma \in \mathbb{R}^{m\times n}$: po diagonali $\sqrt{\lambda_1(A^TA)}, \sqrt{\lambda_2(A^TA)}, \dots, \sqrt{\lambda_n(A^TA)}$
$m \geq n$

(2*) $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times m}$: Neničelne l.vrednosti BC in CB se ujemajo.
($m \geq n$. $BC \in \mathbb{R}^{m\times m}$ ($\mu_1, \mu_2, \dots, \mu_n, 0, \dots, 0$)
$CB \in \mathbb{R}^{n\times n}$ ($\mu_1, \mu_2, \dots, \mu_n$)

(Zakaj? Če $\vec{x} \in \mathbb{R}^m$ l.vekt za BC pri μ: $(BC)\vec{x} = \mu\vec{x}$, $\vec{x} \neq \vec{0}$.

$$\underbrace{(CB)}_{\in\mathbb{R}^{n\times n}}\underbrace{(C\vec{x})}_{\in\mathbb{R}^{n}} = C(BC\vec{x}) = C(\mu\vec{x}) = \mu(C\vec{x})$$

1. Če $C\vec{x} \neq \vec{0}$, potem je l.vekt. CB pri isti l. vrednosti (kot $\vec{x}$ za matriko BC).
2. Če $C\vec{x} = \vec{0} \Rightarrow \underbrace{BC\vec{x}}_{\mu\vec{x}} = \vec{0} \Rightarrow \mu \underbrace{\vec{x}}_{\neq \vec{0}} = \vec{0} \Rightarrow \mu = 0$.

$\Rightarrow$ nenič. l.vrednosti BC so tudi nenič. l.vrednosti CB.

Simetrično: nenič. l.vrednosti CB so tudi nenič. l.vrednosti BC.)

(3) Ali $A = U\Sigma V^T$ en sam? Ne.

Če predpostavimo, da $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & \end{bmatrix}$, $\sigma_1 \geq \dots \geq \sigma_n \geq 0$, je tak Σ en sam. (U in V pa ne nujno.)

Def: Če $A = U\Sigma V^T$, kjer U, V ortogonalni in $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & \end{bmatrix}$ diagonalna, $\sigma_1 \geq \dots \geq \sigma_n \geq 0$, imenujemo $\sigma_1, \dots, \sigma_n$ singularne vrednosti matrike A.

Stolpce matrike U imenujemo levi singularni vektorji A, vrstice V^T (stolpce V) pa desni singularni vektorji A.

Izrek (Razcep singularnih vrednosti):

Za matriko $A \in \mathbb{R}^{m\times n}$ obstajajo:

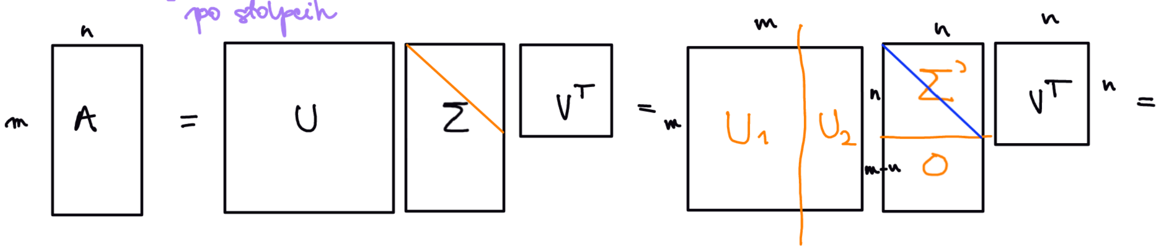
- enolično določena matrika $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & 0 & & \sigma_n \end{bmatrix}$, $\sigma_1 \geq \dots \geq \sigma_n \geq 0$, in
- ortogonalni matriki $U \in \mathbb{R}^{m\times n}$, $V \in \mathbb{R}^{n\times n}$, da velja

$$\boxed{A = U\Sigma V^T.}$$

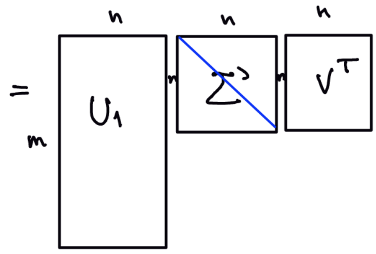
l.vekt. matrike A^TA po vrsticah

$\sqrt{\text{lastnih vrednosti } AA^T}$, če $m \geq n$

l.vekt. matrike AA^T po stolpcih



$= U_1 \Sigma' V^T =$



VARIANTA B: Za $A \in \mathbb{R}^{m\times n}$ obstajajo:

- enolično določena $\Sigma' = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \in \mathbb{R}^{n\times n}$,
- $U_1 \in \mathbb{R}^{m\times n}$ z ortogonalnimi stolpci in
- ortogonalna $V \in \mathbb{R}^{n\times n}$, da velja

$$\boxed{A = U_1 \Sigma' V^T}$$

Če $U_1 = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \in \mathbb{R}^{m\times n}$ in $V^T = \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \in \mathbb{R}^{n\times n}$, potem

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \begin{bmatrix} \longrightarrow v_1^T \longrightarrow \\ \vdots \\ \longrightarrow v_n^T \longrightarrow \end{bmatrix}$$

$$= \begin{bmatrix} \downarrow & \downarrow & & \downarrow \\ u_1 & u_2 & \cdots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \longrightarrow \sigma_1 v_1^T \longrightarrow \\ \longrightarrow \sigma_2 v_2^T \longrightarrow \\ \vdots \\ \longrightarrow \sigma_n v_n^T \longrightarrow \end{bmatrix} = u_1 \underline{\sigma_1} v_1^T + u_2 \underline{\sigma_2} v_2^T + \ldots + u_n \underline{\sigma_n} v_n^T \Rightarrow$$

$$\boxed{A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T}$$

Poseben primer: $A = A^T \in \mathbb{R}^{n\times n}$, naj $\lambda_1, \ldots, \lambda_n$ lastne vrednosti A

$AA^T = A^2 = A^TA$

U ... lastni vekt $AA^T = A^2$ ⟫

V ... lastni vekt $A^TA = A^2$ ⟫

$$\begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} = \underset{U}{\begin{bmatrix} \downarrow & \downarrow \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}} \underset{U^T}{\begin{bmatrix} \longrightarrow \\ \longrightarrow \end{bmatrix}} \quad \text{SVD}$$

singul. vrednosti

l. vektorji A^2

Σ ... $\sqrt{\text{lastne vrednosti } (A^2)} = \{\sqrt{\lambda_1^2}, \sqrt{\lambda_2^2}, \ldots, \sqrt{\lambda_n^2}\} = \{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$

$$\left.\begin{aligned} \lambda_1 + \lambda_2 &= -2 \\ \lambda_1 \lambda_2 &= 1 - 9 = -8 \end{aligned}\right\} \lambda_1 = -4,\ \lambda_2 = 2$$

$$\begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} = \underset{Q}{\begin{bmatrix} \downarrow & \downarrow \end{bmatrix}} \underset{D}{\begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}} \underset{Q^T}{\begin{bmatrix} \longrightarrow \\ \longrightarrow \end{bmatrix}}$$

diagonalizacija sim. matr s spektr. razcepom

l. vektorji A

lastne vrednosti