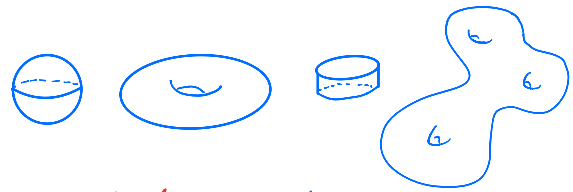
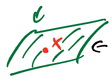
Surfaces



1. Metric setting

Def: $n, d \in \mathbb{N}$, $n \leq d$. A metric space X is an n-manifold if $\forall x \in X \ \exists r_x$:

$B(x, r_x) \cong \mathbb{R}^n \cong B_{\mathbb{R}^n}(0,1)$ or $B(x, r_x) \cong \mathbb{R}^{n-1} \times [0, \infty)$.

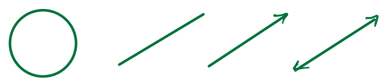

← interior points


← boundary points

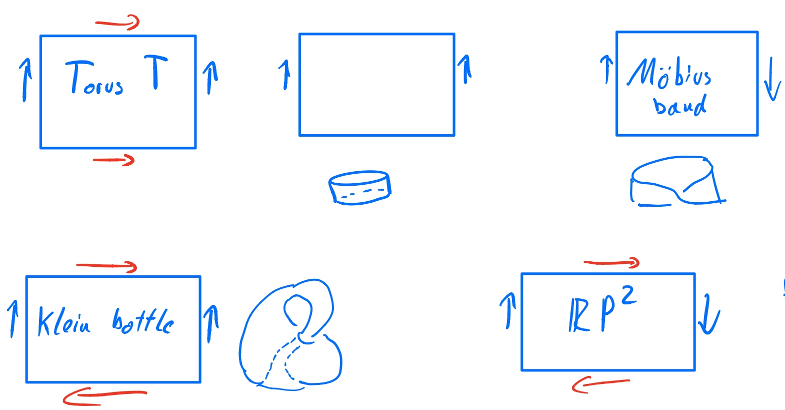
they form the boundary of a manifold: ∂X

surfaces are 2-manifolds.

Examples:
- 1-manifolds

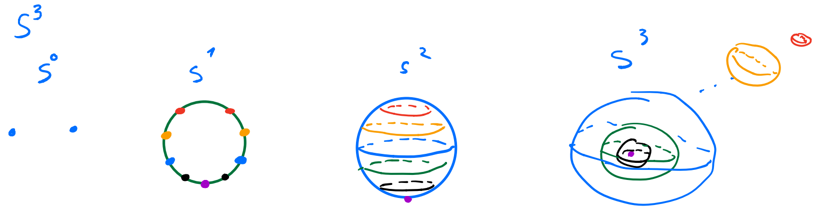


- 2-manifolds (surfaces)



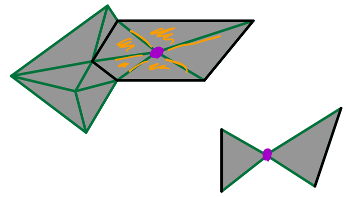
Projective plane

- n-manifolds include S^n, D^n, $\mathbb{R}^n$
- S^3



Remark:
- → A manifold is without boundary if $\partial X = \emptyset$.
- → If X is an n-manifold with a boundary, then ∂X is an $(n-1)$-manifold without boundary. $(\partial\partial X = \emptyset)$.
- → closed manifold: a manifold without boundary admitting a finite triangulation.

2. Combinatorial setting



Def: K scx, $\sigma \in K$

Star $\quad St(\sigma) = \{\tau \in K;\ \sigma \leq \tau\}$ ←

Link $\quad Lk(\sigma) = \{\tau \in K;\ \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in K\} \leq K$

K is a combinatorial n-manifold, if $\forall v \in K^{(0)}$; $Lk(v) \cong S^{n-1}$ or $Lk(v) \cong B^{n-1}$.

Proposition: let K be a comb. n-manifold. Then:

a) $|K| \cong$ n-manifold

b) If $n < 4$, X is an n-manifold $\Rightarrow \exists$ a combinat. n-manifold triangulating X.

---

3. Orientability

Geometric idea:

0-dim $\quad$ a^{+} $\langle a \rangle$ $\quad$ b^{-} $-\langle b \rangle$ $\quad$ charge

1-dim $\quad$ $c \to d$ $\quad$ direction

$\langle c, d \rangle = -\langle d, c \rangle$

2-dim

$\langle e,f,g \rangle = \langle f,g,e \rangle =$
$= \langle g,e,f \rangle =$
$= -\langle e,g,f \rangle = \dots$

σ a sx: $\sigma = \{v_0, v_1, \dots, v_k\}$

$\downarrow$ orient

Def: oriented sx $\langle v_0, v_1, \dots, v_k \rangle$ is an oriented $(k+1)$-tuple:

We identify

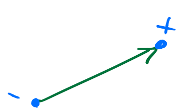
$\langle v_0, v_1, \dots, v_k \rangle = \langle v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(k)} \rangle$ if permutation π on $\{0, 1, \dots n\}$ is even

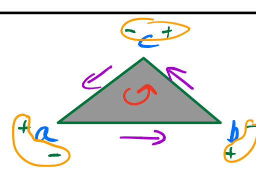
$\langle v_0, v_1, \dots, v_k \rangle = -\langle v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(k)} \rangle$ if permutation π on $\{0, 1, \dots n\}$ is odd

Comments: oriented sxes have sign + or −

oriented vertex b is either $+\langle b \rangle$ or $-\langle b \rangle$

each transposition changes the orientation





$\partial \langle a,b,c \rangle = \langle b,c \rangle - \langle a,c \rangle + \langle a,b \rangle$

$\langle c \rangle - \langle b \rangle + \langle a \rangle - \langle c \rangle + \langle b \rangle - \langle a \rangle = 0$

Important feature: oriented sxes $\underbrace{\text{induce an orientation}}_{\partial}$ on their facets

$$\underset{\uparrow}{\partial}\underbrace{\langle v_0, v_1, \dots, v_k\rangle}_{\sigma} = \sum_{p=0}^{k} (-1)^p v_{\hat p}$$

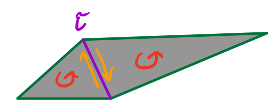
boundary operator.

$v_{\hat p}$ = σ without v_p = $\langle v_0, v_1, \dots, v_{p-1}, v_{p+1}, \dots, v_k\rangle$

Proposition: $\partial \circ \partial = 0$.

Def: K a scx, $\sigma, \sigma' \in K$ oriented n-sxes with a common facet $\tau^{(n-1)}$.

σ and σ' are oriented consistently if the induced orientations on τ are different.

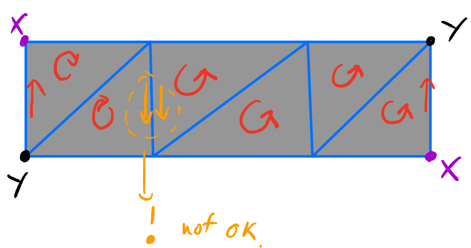


A combinatorial n-manifold is:

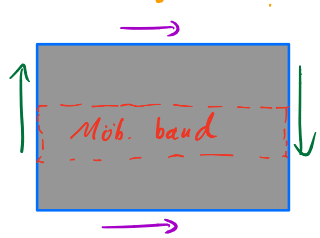
(a) orientable, if its n-sxes CAN be oriented consistently. (homeomorphic invariant.)

(b) oriented, if its n-sxes ARE oriented consistently.

Example:



$\Longrightarrow$ Möbius band not orientable

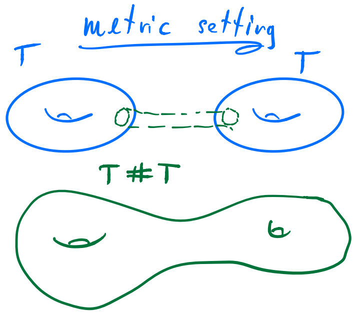


$\Longrightarrow$ Klein bottle ($\mathbb{R}P^2$) not orientable.

---

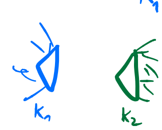
4. Connected sum

metric setting



combinatorial setting

K_1, K_2 comb. surfaces



$-2T\}$ ↳ take a triangle out of each surface

$-3E$, $-3V\}$ ↳ glue the boundaries of the obtained triangles.

$$\chi(K_1 \# K_2) = \chi(K_1) + \chi(K_2) - 2.$$

5. Classification of surfaces

Theorem: K closed connected (comb.) surface. Then K is homeomorphic to one the following:

① S^2

② $\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{g}$ ← genus g surface

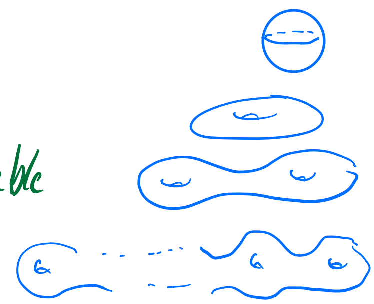
③ $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$

Table:

K	χ	genus g
S^2	2	0
T^2	0	1
$T^2 \# T^2$	-2	2
$\vdots$		
$T^2 \# \cdots \# T^2$	$2-2g$	g
$\mathbb{R}P^2$	1	
$K = \mathbb{R}P^2 \# \mathbb{R}P^2$	0	
$\vdots$		
$\underbrace{\mathbb{R}P^2 \# \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{K}$	$2-K$	

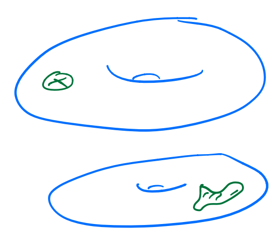
orientable

non-orientable

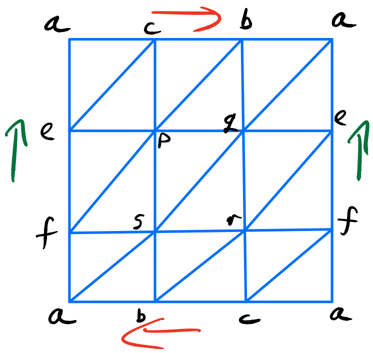


Workflow for a recognition of a ~~closed~~ ~~connected~~ surface:

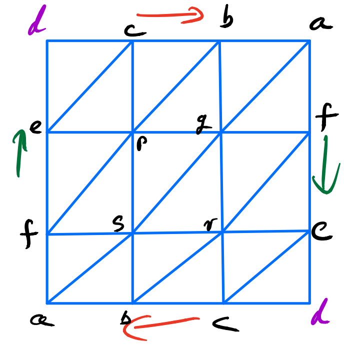
- decompose into components. For each component:
  - attach K-many discs D^2 to boundary
  - compute χ
  - check for orientability
  - consult the table $\rightsquigarrow S$
  - component $\cong S \setminus K$-many D^2 discs.



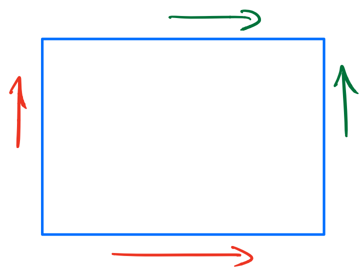
Example:



K $\quad \chi = 0$



$\mathbb{R}P^2 \quad \chi = 1$



Which surface?

HW