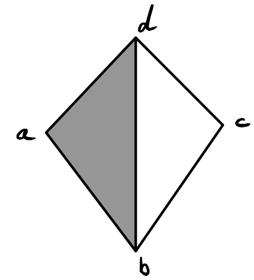
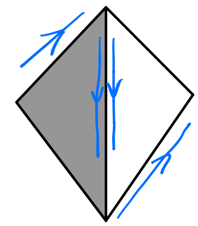
Homology

1. Chains

- K a scx of dim n.
- Choose an Abelian group G for coefficients ($\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{R}$)
- $n_p = \#$ of p-sxes of K, $\forall p \in \{0, 1, \dots, n\}$
- $\forall p \in \{0, 1, \dots, n\}$ a **p-chain** is a formal sum

$$\alpha = \sum_{i=0}^{n_p} \lambda_i \cdot \sigma_i^{(p)} \qquad \lambda_i \in G, \ \underset{\text{ORIENTED!}}{\hat{\sigma_i}} \in K$$

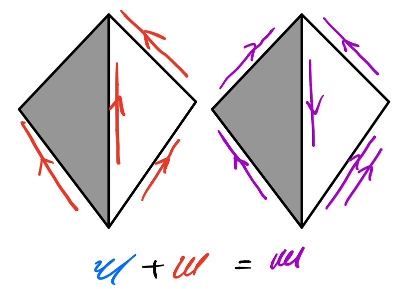




$\langle a,d \rangle +$
$+2\langle d,b \rangle +$
$+\langle b,c \rangle$

- p-chains can be added & subtracted:

$$\sum_{i=0}^{n_p} \lambda_i \sigma_i + \sum_{j=0}^{n_p} \tilde{\lambda}_i \sigma_i = \sum_{i=0}^{n_p} (\lambda_i + \tilde{\lambda}_i) \sigma_i$$



- $C_p(K; G)$ group of p-chains with coef. G.

$$C_p(K; G) \cong G^{n_p}$$

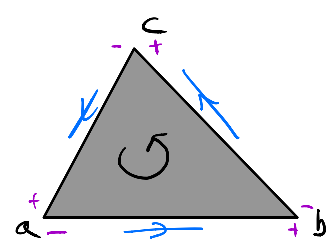
generators / basis: oriented p-sxes.

2. Boundary

- $\sigma = \langle v_0, v_1, \dots, v_p \rangle$ oriented p-sx

boundary of σ $\quad \partial \sigma \in C_{p-1}(K; G)$

$$\partial \sigma = \sum_{i=0}^{p} (-1)^i \underbrace{\langle v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p \rangle}_{\text{drop } v_i \text{ from } \sigma}$$



$\partial \langle a,b,c \rangle = \langle a,b \rangle + \langle b,c \rangle + \langle c,a \rangle$

$\partial^2 \langle a,b,c \rangle = 0$

- This induces a map (homomorphism / linear map)

$$\partial_p : C_p(K; G) \to C_{p-1}(K; G)$$

$$\sum \lambda_i \sigma_i \mapsto \sum \lambda_i \cdot \partial \sigma_i$$

- chain complex:

$$\cdots \to C_2(K; G) \xrightarrow{\partial_2} C_1(K; G) \xrightarrow{\partial_1} C_0(K; G) \xrightarrow{\partial_0 = 0} 0$$

THM: $\partial^2 = 0 \qquad (\forall p \geq 0: \ \partial_p \circ \partial_{p+1} = 0)$.

Proof:

$\sigma = \langle v_0, v_1, \dots, v_p \rangle$

drop v_i first (∂)

drop v_j later (∂)

∂^2

drop v_j first

drop v_i then

$$\underbrace{\left((-1)^i(-1)^{j-1} + (-1)^j(-1)^i\right)}_{0} \underbrace{\langle v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_p \rangle}_{\text{dropped } v_i \,\&\, v_j \text{ from } \sigma}$$

$\blacksquare$

Corollary: $\operatorname{Im} \partial_p \subseteq \operatorname{Ker} \partial_{p-1}$.

3. Homology

$\forall p \in \{0, 1, \dots, n\}$ define two subgroups of $C_p(K; G)$

p-cycles $\quad Z_p(K; G) = \operatorname{Ker} \partial_p \quad \leftarrow$ potential repres. of p-holes

p-boundaries $\quad B_p(K; G) = \operatorname{Im} \partial_{p+1} \quad \leftarrow$ cycles representing trivial holes

p-homology group

$$H_p(K; G) = Z_p(K; G) / B_p(K; G)$$

$\leftarrow$ holes

$\leftarrow$ elements: equivalence classes of p-cycles

THM: $K \simeq L \Rightarrow H_*(K; G) \cong H_*(L; G), \quad \forall G$

Homology is a homotopy invariant (as opposed to Z_p, B_p, C_p)

Homology is of the following form:

Betti numbers

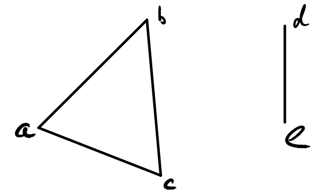
$H_p(K; \mathbb{R}) \cong \mathbb{R}^{D_p}$ $\qquad b_p(K; \mathbb{R}) = D_p$

$H_p(K; \mathbb{Z}_p) \cong \mathbb{Z}_p^{d_p}$ $\qquad \leftarrow$ p a prime number $\quad b_p(K; \mathbb{Z}_p) = d_p$

$H_p(K; \mathbb{Z}) \cong \underbrace{\mathbb{Z}^{r_p}}_{\text{free part}} \oplus \underbrace{(\mathbb{Z}_{q_1} \oplus \dots \oplus \mathbb{Z}_{q_k})}_{\text{torsion}}$ $\qquad b_p(K; \mathbb{Z}) = r_p$

ranks of homology groups, "count" the number of p-holes in K.

Example: H_0 with coefficients in G.



$$\begin{array}{ccccc} C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\ \| \wr & & \| \wr & & \\ G^4 & & G^5 & & \end{array}$$

G^5 generated by $\langle a\rangle, \langle b\rangle, \dots, \langle e\rangle$

G^4 generated by $\langle a,b\rangle, \langle b,c\rangle, \langle a,c\rangle, \langle d,e\rangle$

$$H_0 = \frac{\operatorname{Ker} \partial_0}{\operatorname{Im} \partial_1} \qquad \operatorname{Ker} \partial_0 = C_0$$

$\partial\langle a,b\rangle = \langle b\rangle - \langle a\rangle$

$[\langle b\rangle] - [\langle a\rangle] = 0$

$[\langle b\rangle] = [\langle a\rangle]$

$\partial\langle b,c\rangle = \langle c\rangle - \langle b\rangle$

$\partial\langle a,c\rangle = \langle c\rangle - \langle a\rangle$ redundant

$\partial\langle d,e\rangle = \langle e\rangle - \langle d\rangle$

generators of H_0:

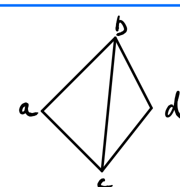
$[\langle a\rangle] = [\langle b\rangle] = [\langle c\rangle]$

$[\langle d\rangle] = [\langle e\rangle]$

\# of components (0-holes)

Proposition: $H_0(K;G) \cong G^{\#\text{ of components}}$

Example: H_1 with coefficients in $\mathbb{Z}_2$



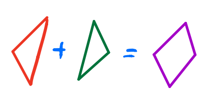
2 holes

$$\begin{array}{ccccc} C_2(K;\mathbb{Z}_2) & \longrightarrow & C_1(K;\mathbb{Z}_2) & \longrightarrow & C_0(K;\mathbb{Z}_2) \\ \| & & \| & & \| \\ 0 & & \mathbb{Z}_2^5 & & \mathbb{Z}_2^4 \end{array}$$

$$H_1(K;\mathbb{Z}_2) = \operatorname{Ker} \partial_1 / \operatorname{Im} \partial_2 = \operatorname{Ker} \partial_1$$

4 elements

$\mathbb{Z}_2^{2}$ $\to$ 2 holes

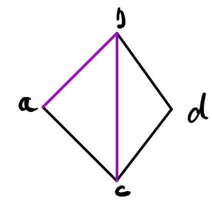


3 non-trivial 1-cycles + 1 trivial 1-cycle

Example: H_1 with coefficients in G

K a planar graph.

$$\begin{array}{ccccc} C_2(K;G) & \longrightarrow & C_1(K;G) & \longrightarrow & C_0(K;G) \\ \| & & \| & & \| \\ 0 & & G^5 & & G^4 \end{array}$$



$$H_1(K;G) = {\mathrm{Ker}\,\partial_1}/{\mathrm{Im}\,\partial_2} = \mathrm{Ker}\,\partial_1$$

basis of C_1:

	$\langle a,b\rangle$	$\langle b,d\rangle$	$\langle d,c\rangle$	$\langle a,c\rangle$	$\langle b,c\rangle$
boundaries	$\langle b\rangle - \langle a\rangle$	$\langle d\rangle - \langle b\rangle$	$\langle c\rangle - \langle d\rangle$	$\langle c\rangle - \langle a\rangle$	$\langle c\rangle - \langle b\rangle$
	obsolete				obsolete
					$\underbrace{\langle c\rangle - \langle d\rangle}_{3^{rd}} + \underbrace{\langle d\rangle - \langle b\rangle}_{2^{nd}}$

$$-\underbrace{(\langle d\rangle - \langle b\rangle)}_{2^{nd}} - \underbrace{(\langle c\rangle - \langle d\rangle)}_{3^{rd}} + \underbrace{\langle c\rangle - a\rangle}_{4^{th}}$$

"$\mathrm{Ker}\,\partial_1$" is generated by 2 elements $\rightsquigarrow$ 2 holes

Proposition: K a planar graph $\Rightarrow H_1(K;G) \cong G^{\#\text{ of holes}}$

Computing homology with matrix reductions

choose a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{Z}_p \text{ for } p \text{ prime}\}$

K a scx of dim n. Fix $q \in \{0,1,\dots,n\}$

$$H_q(K;\mathbb{F}) \cong \mathbb{F}^{r_q} \cong {\mathrm{Ker}\,\partial_q}/{\mathrm{Im}\,\partial_{q+1}}$$

Proposition: $f: A \to B$ linear map of vector spaces, $\mathrm{rank} f = \dim f(A)$. Then:

(i) $\dim A = \dim \mathrm{Ker} f + \mathrm{rank} f$

(ii) $\dim\left(B/f(A)\right) = \dim B - \mathrm{rank} f$

Corollary: ⓐ $\dim \mathrm{Ker}\,\partial_q = \dim C_q - \mathrm{rank}\,\partial_q =$

$= (\#\text{ of } q\text{-sxes}) - \mathrm{rank}\,\partial_q$

ⓑ $r_q = (\#\text{ of } q\text{-sxes}) - \mathrm{rank}\,\partial_q - \mathrm{rank}\,\partial_{q+1}.$

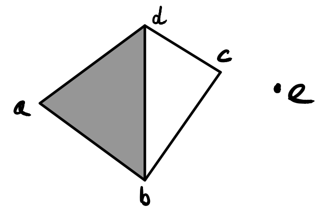
Gauss,...

How do we compute ranks: $\partial_q \rightsquigarrow$ Matrix $\rightsquigarrow$ rank

The nicest case: SMITH NORMAL FORM

$$\begin{array}{cc} & \text{gen.}\ \overbrace{\quad\quad}^{\mathrm{Im}\,\partial_q}\ \overbrace{\quad\quad\quad\quad}^{\mathrm{Ker}\,\partial_q} \\ \mathrm{rank}\,\partial_q \left\{ \right. & \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & & & \\ & & & & & \end{array}\right] \end{array}$$

Example: H_* with coef. in a field $\mathbb{F}$



$$\underset{\overset{\shortparallel}{0}}{C_3} \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

- rank $\partial_0 = 0$

- ∂_1

$$\begin{array}{c} \\ \langle a\rangle \\ \langle b\rangle \\ \langle c\rangle \\ \langle d\rangle \\ \langle e\rangle \end{array}
\begin{array}{c}
\begin{array}{ccccc} \langle a,b\rangle & \langle b,c\rangle & \langle c,d\rangle & \langle a,d\rangle & \langle b,d\rangle \end{array} \\
\left[\begin{array}{cc|c|cc}
-1 & & & -1 & \\
1 & -1 & & & -1 \\ \hline
 & 1 & -1 & & \\ \hline
 & & 1 & 1 & 1 \\
 & & & &
\end{array}\right]
\end{array}
\rightsquigarrow \text{Gauss. through rows}$$

$$\left[\begin{array}{cc|c|cc}
-1 & & & -1 & \\
 & -1 & & -1 & -1 \\ \hline
 & & -1 & -1 & -1 \\ \hline
 & & & & \\
 & & & &
\end{array}\right] \rightsquigarrow \text{rank } \partial_1 = 3$$

- ∂_2

$$\begin{array}{c} \\ \langle a,b\rangle \\ \langle b,c\rangle \\ \langle c,d\rangle \\ \langle a,d\rangle \\ \langle b,d\rangle \end{array}
\begin{array}{c} \langle a,b,d\rangle \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{array}\right] \end{array}
\qquad \text{rank } \partial_2 = 1$$

- ∂_3 trivial

Conclusion: $r_0 = 5 - 0 - 3 = 2$ components

$r_1 = 5 - 3 - 1 = 1$ hole

$r_2 = 1 - 1 - 0 = 0$ caves / voids

$\boxed{\text{Week 8}}$

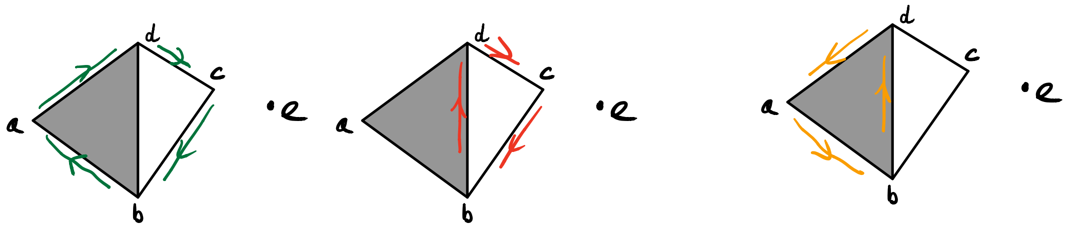
How to get representatives?

$$\begin{array}{c}
\begin{array}{ccccc} \langle a,b\rangle & \langle b,c\rangle & \langle c,d\rangle & \langle a,d\rangle & \langle b,d\rangle \end{array} \\
\left[\begin{array}{c|c|c|cc}
-1 & & & -1 & \\
 & -1 & & -1 & -1 \\ \hline
 & & -1 & -1 & -1 \\
 & & & &
\end{array}\right]
\end{array}
\rightsquigarrow
\begin{array}{c}
\begin{array}{ccccc} \langle a,b\rangle & \langle b,c\rangle & \langle c,d\rangle & \alpha & \beta \end{array} \\
\left[\begin{array}{ccc|cc}
-1 & & & & \\
 & -1 & & & \\
 & & -1 & & \\ \hline
 & & & &
\end{array}\right]
\end{array}$$

$\alpha = \langle a,d\rangle - \langle a,b\rangle - \langle b,c\rangle - \langle c,d\rangle$

$\beta = \langle b,d\rangle - \langle b,c\rangle - \langle c,d\rangle$

$\underbrace{\alpha \quad \beta}_{\text{basis of } Z_1}$



What is a basis of $\operatorname{Im} \partial_2$? $\langle a,b\rangle + \langle b,d\rangle + \langle d,a\rangle$ ← basis of B_1

$$H_1 = Z_1 / B_1$$

Basis of H_1? ① Take a basis of B_1. ← of dim 1

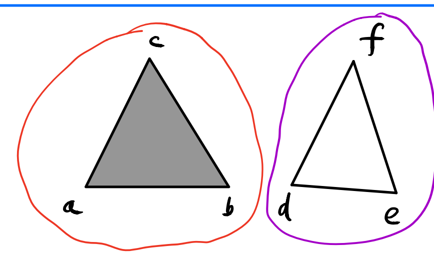
② Complete it to the basis of Z_1. ← of dim 2

③ $\underbrace{\text{The added cycles}}$ form the basis of H_1.

for example, β, or α.

Example: coefficients in a field $\mathbb{F}$

$$\begin{array}{ccccccccc} C_3 & \xrightarrow{0} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 \\ \cong & & \cong & & \cong & & \cong & & \\ 0 & & \mathbb{F}^1 & & \mathbb{F}^6 & & \mathbb{F}^6 & & \end{array}$$



$$\partial_2: \begin{array}{c|c} & \langle a,b,c\rangle \\ \hline \langle a,b\rangle & 1 \\ \langle b,c\rangle & 1 \\ \langle c,a\rangle & 1 \\ \langle d,e\rangle & 0 \\ \langle e,f\rangle & 0 \\ \langle f,d\rangle & 0 \end{array}$$

$$\partial_1: \begin{array}{c|cccccc} & \langle a,b\rangle & \langle b,c\rangle & \langle c,a\rangle & \langle d,e\rangle & \langle e,f\rangle & \langle f,d\rangle \\ \hline \langle a\rangle & -1 & & 1 & & & \\ \langle b\rangle & 1 & -1 & & & & \\ \langle c\rangle & & 1 & -1 & & & \\ \langle d\rangle & & & & -1 & & 1 \\ \langle e\rangle & & & & 1 & -1 & \\ \langle f\rangle & & & & & 1 & -1 \end{array}$$

$\partial_0 : 0$

$\text{rank}_2 = 1$ $\qquad$ $\text{rank}_1 = 4$ $\qquad$ $\text{rank}_0 = 0$

$b_0 = 6 - 0 - 4 = 2$ components

$b_1 = 6 - 4 - 1 = 1$ hole

$b_2 = 1 - 1 - 0 = 0$ void

Observation: K, L scx, $K \cap L = \emptyset \Rightarrow H_*(\underbrace{K \cup L}_{\text{disjoint union}}) = H_*(K) \oplus H_*(L)$

for any coefficients.

The Euler characteristic: K^m scx, n_p = # of p-sxes of K, b_p = p^{TH} Betti number, for any coefficients.

Theorem: $\chi = n_0 - n_1 + n_2 - \dots = b_0 - b_1 + b_2 - \dots$

Proof:

$$b_0 = n_0 - \text{rank}\,\partial_0 - \text{rank}\,\partial_1$$

$$b_1 = n_1 - \text{rank}\,\partial_1 - \text{rank}\,\partial_2$$

$$\vdots$$

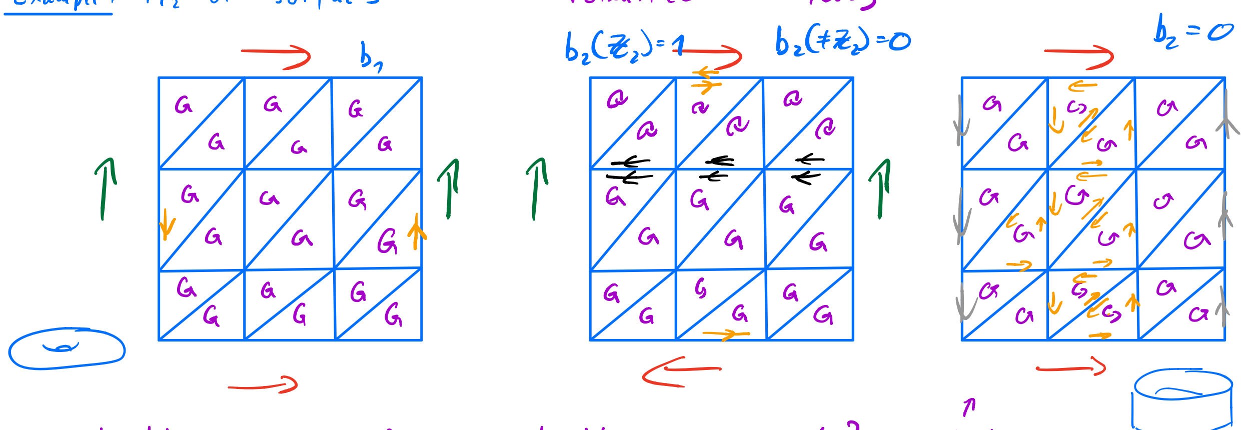
$$b_m = n_m - \text{rank}\,\partial_m - \underbrace{\text{rank}\,\partial_{m+1}}_{=0}$$

∎

Corollary: K a planar scx $\Rightarrow \chi(K)$ = # of components − # of holes.

Example: H_2 of surfaces

Potential 2-cycles



Is this a 2-cycle?
Yes it is, $b_2 = 1$

Is this a 2-cycle?
Its boundary: $2\cdot\alpha$
if $G=\mathbb{Z}_2$ → 0
non-zero if $G \neq \mathbb{Z}_2$

is this a 2-cycle?
No, its boundary not 0.

Theorem: K a connected combinatorial surface.

① If $\partial K \neq 0 \Rightarrow H_2(K;G) = 0$, $\forall G$.

② If $\partial K = 0$:

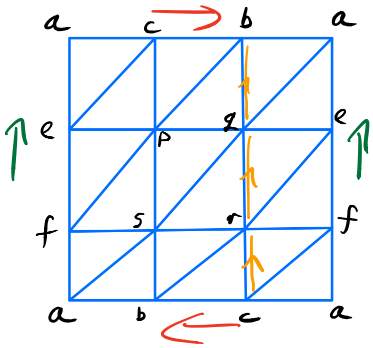
ⓐ If K orientable, $H_2(K;G) \cong G$ → generator $= [\Sigma(\text{consist. oriented } \Delta\text{'s})]$ fundamental class

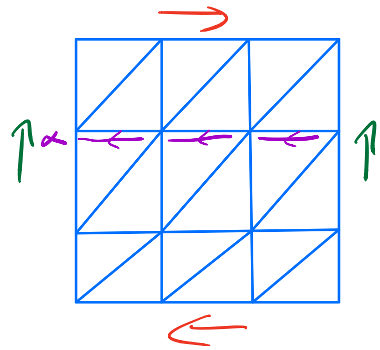
ⓑ If K non-orientable, $H_2(K;\mathbb{Z}_2) \cong \mathbb{Z}_2$

$H_2(K;G) = 0$ for $G \not\equiv \mathbb{Z}_2$.

[THM holds for manifolds of any dimension]

Example of torsion: The Klein bottle K





$$\underbrace{[2\alpha]}_{\text{it is a boundary!}} = 0$$

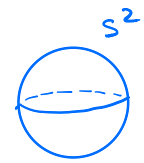
$$H_1(K;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

$$H_1(K;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$H_1(K;\mathbb{Z}_p) \cong \mathbb{Z}_p \quad \text{if } p \neq 2.$$

Example: Spheres

$$H_p(S^n;G) \cong \begin{cases} G & \text{for } p \in \{0,n\} \\ 0 & \text{else} \end{cases}$$



$b_0 = 1$
$b_1 = 0$
$b_2 = 1$

Proof: $S^n = \partial B^{n+1}$ $\qquad B^{n+1} \simeq \bullet \Rightarrow H_p(B^{n+1};G) = \begin{cases} G & p=0 \\ 0 & \text{else} \end{cases}$

chain cx of B^{n+1}

$$\begin{array}{ccccccccc} C_{n+2}(B^{n+1}) & \to & C_{n+1}(B^{n+1}) & \xrightarrow{\text{rank } 1} & C_n(B^{n+1}) & \to & C_{n-1}(B^{n+1}) & \to \cdots \to & C_0 \\ \cong & & \cong & & \cong & & & & \\ 0 & & G^1 & & G^{n+2} & & & & \end{array}$$

remove the single $(n+1)$-sx

$$\begin{array}{ccccccccc} C_{n+2}(S^n) & \to & C_{n+1}(S^n) & \xrightarrow{\text{rank } 0} & C_n(S^n) & \to & C_{n-1}(S^n) & \to \cdots \to & C_0 \\ \cong & & \cong & & \cong & & & & \\ 0 & & G^0 & & G^{n+2} & & & & \end{array}$$

$$\Rightarrow H_n(S^n;G) \cong G. \qquad \blacksquare$$

Induced maps: K, L scx, $m \in \{0,1,\dots\}$, G coefficients

$f: K \to L$ simplicial map $\rightsquigarrow$ Induced homomorphism $f_*: H_m(K;G) \to H_m(L;G)$
(linear map)

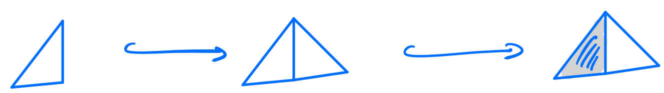
$$f_*([\sum \lambda_i \sigma_i]) = [\sum \lambda_i \; f(\sigma_i)]$$

This turns out to be well defined.

Functoriality: M scx, $g: L \to M$ simplicial, then

$$(g \circ f)_* = g_* \circ f_*$$

Example:

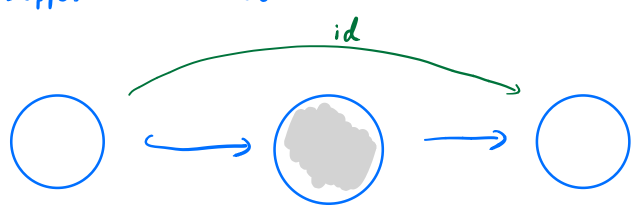


Induced maps on $H_1(-;\mathbb{Z})$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z} \\ & (n,m) \arrow[r, mapsto] & m \\ n \arrow[r, mapsto] \arrow[rr, bend right, mapsto] & (n,0) \arrow[r, mapsto] & 0 \end{tikzcd}$$

Example: Does there exist a continuous retraction $f: B^2 \to S^1$? NO.

Suppose it can be done. $x \in S^1 \Rightarrow f(x) = x$.



$\Big\downarrow H_1(-;\mathbb{Z}) \Big\downarrow$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend left, "id"] & 0 \arrow[r] & \mathbb{Z} \\ n \arrow[r, mapsto] \arrow[rr, bend left, mapsto] & 0 \arrow[r, mapsto] & 0 \end{tikzcd}$$

contradiction, f does not exist. ∎

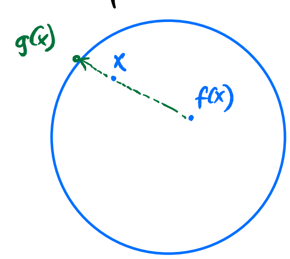
Brouwer's fixed point theorem: $f: D^n \to D^n$ continuous $\Rightarrow$ f has a fixed point, i.e. $x_0 \in D^n$: $f(x_0) = x_0$.

Proof: $n=2$, f simplicial.

Assume such f exists.

Then there exists a retraction

$g: D^2 \to S^1$, a contradiction. $\Rightarrow\Leftarrow$

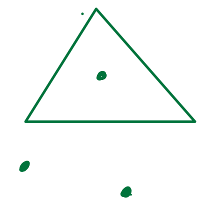


∎

Alexander duality

$K \subset \mathbb{R}^2$ scx

$$b_0(K) = b_1(K^c)$$
$$b_1(K) = b_0(K^c) - 1$$



$L \subset \mathbb{R}^3$ scx

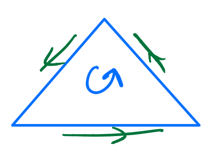
$$b_2(L) = b_0(L^c)$$
$$b_1(L) = b_1(L^c)$$

"Complexes" can be constructed using many different shapes as building blocks.

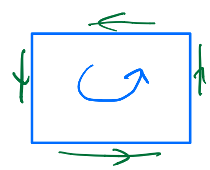
Cubical complexes $\rightsquigarrow$ assembled from cubes

define cubical homology in the same way using the boundary operator.

simplicial boundary



cubical boundary



$\partial^2 = 0$

homology.