Definicija: Za matriko $A \in \mathbb{R}^{m \times n}$ imenujemo

$$N(A) = \{\vec{x} \in \mathbb{R}^n;\ A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n$$

ničelni prostor matrike A.
(null space)

$N(A)$ je vekt. podprostor v $\mathbb{R}^n$

Primer: $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ -1 & -1 & -2 & -1 \\ 2 & 2 & 4 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 4}$. Izračunajmo $N(A)$.

$\equiv$ rešujemo homogeni sistem $A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ -1 & -1 & -2 & -1 \\ 2 & 2 & 4 & 2 \end{bmatrix} \xrightarrow{G.e.} \begin{array}{cccc} x & y & z & w \end{array} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow x + y + 2z + w = 0$$

$/\cdot(-2)$, $+$, $+$

PROSTE SPR.

$x = -y - 2z - w$

$v_2 \leftarrow v_2 + v_1$
$v_3 \leftarrow v_3 - 2v_1$

rang $= 1$

$$N(A) = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix};\ x + y + 2z + w = 0 \right\} = \left\{ \begin{bmatrix} -y-2z-w \\ y \\ z \\ w \end{bmatrix};\ y, z, w \in \mathbb{R} \right\} =$$

$$= \left\{ y \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix};\ y, z, w \in \mathbb{R} \right\} =$$

$$= \mathcal{L}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3.3 BAZE VEKTORSKIH PROSTOROV

V vektorski prostor

Def: Vektorji $v_1, \dots, v_k \in V$ so linearno odvisni, če lahko kakšnega izmed njih izrazimo kot linearno kombinacijo ostalih:

obstaja $j \in \{1, \dots, k\}$: $v_j = \alpha_1 v_1 + \dots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \dots + \alpha_k v_k$

Vektorji so linearno neodvisni, če niso linearno odvisni.

Če $v_1, \dots, v_k$ lin. odvisni: $\alpha_1 v_1 + \dots + \alpha_{j-1} v_{j-1} + (-1) v_j + \alpha_{j+1} v_{j+1} + \dots + \alpha_k v_k = \vec{0}$ $\quad (-1) \neq 0$

NISO VSI SKALARJI ENAKI 0

$\Rightarrow$ neka netrivialna linearna kombinacija vektorjev $v_1, \dots, v_k$ je enaka $\vec{0}$.

Lin. neodvisnost: edina lin. kombinacija $v_1, \dots, v_k$, ki je enaka $\vec{0}$ je tista z ničelnimi koeficienti.

Če $\alpha_1 v_1 + \dots + \alpha_k v_k = \vec{0} \Rightarrow \alpha_1 = \dots = \alpha_k = 0$

Primer: ① Ali so $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ in $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ linearno neodvisne?

Če $\alpha \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, potem je

$$\begin{bmatrix} \alpha + \beta & 2\alpha + \beta + \gamma \\ \beta + \gamma & 3\alpha + \beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \alpha + \beta &= 0 \\ 3\alpha + \beta &= 0 \end{aligned} \quad] -$$

$$\beta + \gamma = 0$$

$$2\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow \beta = 0 \Rightarrow \gamma = 0.$$

$\Rightarrow$ Da, so lin. neodvisne.

② Ali so matrike $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ in $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ lin. neodvisne?

Če $\alpha \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{aligned} \alpha + \beta &= 0 \\ -\beta + \gamma &= 0 \\ 2\alpha + \beta + \gamma &= 0 \\ \cancel{3\alpha + 3\beta} &\cancel{= 0} \end{aligned} \quad \begin{aligned} &\Rightarrow \alpha = -\beta \\ &\Rightarrow \gamma = \beta \end{aligned} \quad 2(-\beta) + \beta + \beta = 0, \quad 0 = 0 \ ✓$$

Naj bo $\beta = 1 \Rightarrow \alpha = -1, \gamma = 1$:

$$-\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ NE! So lin. odvisne.

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

③ $\vec{a}=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b}=\begin{bmatrix}2\\4\\6\end{bmatrix}$, $\vec{c}=\begin{bmatrix}0\\0\\1\end{bmatrix}$ Ali so $\vec{a},\vec{b},\vec{c}$ lin. neodvisni?

Opazimo: $\left.\begin{array}{l}\vec{b}=2\vec{a}=2\vec{a}+0\cdot\vec{c}\\ \vec{a}=\frac{1}{2}\vec{b}+0\cdot\vec{c}\end{array}\right\}$ NE. $\vec{b}$ je lin. komb. $\vec{a}$ in $\vec{c}$.

Ampak: Če $\vec{c}=\alpha\vec{a}+\beta\vec{b}=\alpha\vec{a}+\beta\cdot 2\vec{a}=(\alpha+2\beta)\vec{a}$ $\rightarrow\leftarrow$

Def: Množica $\mathcal{B}=\{b_1,\dots,b_k\}\subseteq V$ je baza vekt. pr. V, če velja

(1) $\mathcal{L}\{b_1,\dots,b_k\}=V$ → vsak vekt. iz V je lin. komb. vekt. $\{b_1,\dots,b_k\}$

→ jih je dovolj veliko

(2) $b_1,\dots,b_k$ so linearno neodvisni

→ jih ni preveč

Primer: Določimo bazo $\mathcal{A}=\left\{\begin{bmatrix}x\\y\\-y\\-x\end{bmatrix};\ x,y\in\mathbb{R}\right\}\subseteq\mathbb{R}^4$

$$\begin{bmatrix}x\\y\\-y\\-x\end{bmatrix}=x\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}+y\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\Rightarrow \mathcal{A}=\mathcal{L}\left\{\underset{\vec{a}}{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}},\underset{\vec{b}}{\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}}\right\}$$

vsak vekt iz $\mathcal{A}$ je lin. komb. $\vec{a}$ in $\vec{b}$

Ali sta $\vec{a}$ in $\vec{b}$ lin. neodvisna?

$$\alpha\vec{a}+\beta\vec{b}=\vec{0}$$

$$\begin{bmatrix}\alpha\\\beta\\-\beta\\-\alpha\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\Rightarrow\alpha=\beta=0\Rightarrow\vec{a}\text{ in }\vec{b}\text{ lin. neodvisna.}$$

$\Rightarrow \mathcal{A}=\mathcal{L}\{\vec{a},\vec{b}\}$ in $\{\vec{a},\vec{b}\}$ lin. neodv.

$\Rightarrow \{\vec{a},\vec{b}\}$ je baza $\mathcal{A}$. ← $\dim\mathcal{A}=2$

Lastnosti: 1) Vsak vektorski prostor ima neskončno baz.
Vse baze imajo enako število vektorjev.
To število imenujemo dimenzija prostora: $\dim(V)$.

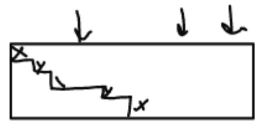
Zakaj? Denimo, da $\underbrace{\{b_1, \dots, b_n\}}_{= \mathcal{B}}$ in $\underbrace{\{c_1, \dots, c_m\}}_{= \mathcal{C}}$ bazi V, $\boxed{m > n}$.

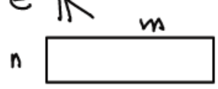
Ker $\mathcal{B}$ baza

dovolj: $\mathcal{L}\{b_1, \dots, b_n\} = V$

$$c_1 = \alpha_{11} b_1 + \alpha_{12} b_2 + \dots + \alpha_{1n} b_n$$
$$c_2 = \alpha_{21} b_1 + \dots + \alpha_{2n} b_n$$
$$\vdots$$
$$c_m = \alpha_{m1} b_1 + \dots + \alpha_{mn} b_n$$

$$[c_1\ c_2\ \cdots\ c_m] = [b_1\ b_2\ \cdots\ b_n] \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{21} & & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{m2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{mn} \end{bmatrix}}_{= A \in \mathbb{R}^{n \times m}}$$

$A \xrightarrow{G.e.}$ 



$\Rightarrow A\vec{x} = \vec{0}$ ima rešitev $\vec{x}_0 = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} \neq \vec{0}$

$$[c_1 \cdots c_m] = [b_1 \cdots b_n] \cdot \underline{A} \quad / \underline{\vec{x}_0} \quad (A\vec{x}_0 = \vec{0})$$

$$[c_1 \cdots c_m] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = \vec{0}$$

$$\beta_1 c_1 + \beta_2 c_2 + \dots + \beta_m c_m = \vec{0}$$ (ker $c_1, \dots, c_m$ lin. neodvisni)

$c_1, \dots, c_m$ lin. neodvisni $\rightarrow$

$\Rightarrow \beta_1 = \beta_2 = \dots = \beta_m = 0$

protislovje

$\Rightarrow m \leq n$.

Podobno pokažemo, da $n \leq m$. $\Big\}\ m = n$

2) $\dim V$ je
- največje število lin. neodvisnih vektorjev
- najmanjše število vektorjev, ki razpenjajo V.

3) Če $\{b_1, \dots, b_k\}$ baza V, potem se vsak $v \in V$ na enoličen način izrazi kot lin. komb.

$$v = \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_k b_k.$$

Zakaj? Če $v = \beta_1 b_1 + \dots + \beta_k b_k = \gamma_1 b_1 + \dots + \gamma_k b_k$

$$\beta_1 b_1 + \dots + \beta_k b_k - \gamma_1 b_1 - \dots - \gamma_k b_k = 0$$

$b_1, \dots, b_k$ lin. neodv. $\longrightarrow$ $(\beta_1 - \gamma_1) b_1 + (\beta_2 - \gamma_2) b_2 + \dots (\beta_k - \gamma_k) b_k = 0$

$$\begin{aligned} \beta_1 - \gamma_1 &= 0 \quad &&\Rightarrow \beta_1 = \gamma_1 \\ \beta_2 - \gamma_2 &= 0 \quad &&\Rightarrow \beta_2 = \gamma_2 \\ &\vdots && \vdots \\ \beta_k - \gamma_k &= 0 \quad &&\Rightarrow \beta_k = \gamma_k \end{aligned}$$

$\Rightarrow$ en sam zapis kot lin. komb. $b_1, \dots, b_k$.

Primer: Vektorski podprostori v $\mathbb{R}^3$:

- $\dim U = 3 \Rightarrow U = \mathbb{R}^3$
- $\dim U = 2 \Rightarrow U$ je ravnina (skozi koord. izh.)
  $U = \mathcal{L}\{\vec{a}, \vec{b}\}$, $\vec{a}, \vec{b}$ lin. neodvisna
- $\dim U = 1 \Rightarrow U$ je premica (skozi koord. izh.)
  $U = \mathcal{L}(\vec{a})$
- $\dim U = 0 \Rightarrow U = \{\vec{0}\}$ $\equiv$ trivialni vekt. pr.

Primer: 1 V $\mathbb{R}^n$ označimo

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\{\vec{e}_1, \dots, \vec{e}_n\}$ je baza $\mathbb{R}^n$

- $\mathcal{L}\{\vec{e}_1, \dots, \vec{e}_n\} = \{\alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_n \vec{e}_n;\ \alpha_i \in \mathbb{R}\} =$

$$= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}; \alpha_i \in \mathbb{R} \right\}$$

$\dim \mathbb{R}^n = n$

- $\vec{e}_1, \dots, \vec{e}_n$ so lin. neodvisni vektorji
  (če $\alpha_1 \vec{e}_1 + \dots + \alpha_n \vec{e}_n = \vec{0} \Rightarrow \alpha_1 = \dots \alpha_n = 0$)

$\Rightarrow \{\vec{e}_1, \dots, \vec{e}_n\}$ baza $\mathbb{R}^n$ $\leftarrow$ standardna baza $\mathbb{R}^n$

② V $\mathbb{R}^{m\times n}$:
$$E_{ij} = \begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} & \\ & 1 \end{bmatrix} \end{array} \in \mathbb{R}^{m\times n}$$

DN: $\{E_{11}, E_{12}, \dots, E_{1n}, E_{21}, E_{22}, \dots, E_{2n}, \dots, E_{m1}, \dots, E_{mn}\}$ je baza $\mathbb{R}^{m\times n}$ ← standardna baza $\mathbb{R}^{m\times n}$

$\dim \mathbb{R}^{m\times n} = mn$

TRIK: Pri Gaussovi eliminaciji se linearna ogrinjača ne spreminja:

(1) $\mathcal{L}\{u,v\} = \mathcal{L}\{v,u\}$ jasno

(2) $\mathcal{L}\{u, v_2, \dots, v_k\} = \mathcal{L}\{\alpha u, v_2, \dots, v_k\}$ jasno

(3) $\mathcal{L}\{u, v, v_3, \dots, v_k\} = \mathcal{L}\{u, v+\alpha u, v_3, \dots, v_k\}$ (napišite na dolgo sami)

Def: Stolpčni prostor matrike $A \in \mathbb{R}^{m\times n}$ je lin. ogrinjača njenih stolpcev. Označimo ga s $C(A)$. Je vekt. podpr. v $\mathbb{R}^m$.

(column space)

Primer: a) Določimo $C(A)$ matrike $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}$.

$$C(A) = \mathcal{L}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\} = \left\{\alpha\begin{bmatrix}1\\0\\1\end{bmatrix} + \beta\begin{bmatrix}1\\1\\0\end{bmatrix} + \gamma\begin{bmatrix}2\\1\\1\end{bmatrix} + \delta\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$$

b) Določimo bazo $C(A)$ iz a).

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\cdot(-2)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \operatorname{rang} A = 2 \quad \}\ \text{lin. neodvisni}$$

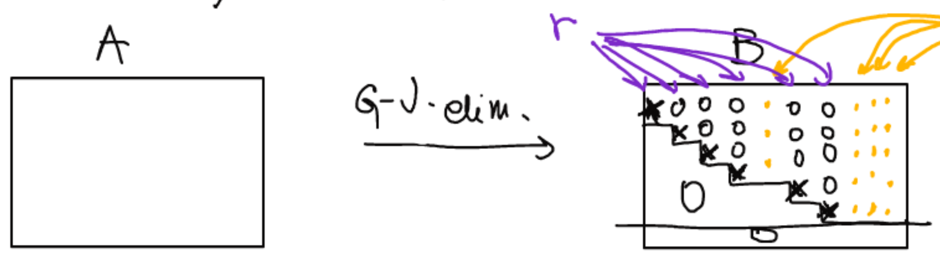
$\mathcal{L}$

$$C(A) = \mathcal{L}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$$

lin. neodv.

$$\Rightarrow \text{baza } C(A) = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}, \quad \dim C(A) = 2$$

$A \in \mathbb{R}^{m \times n}$, kakšni sta dim $N(A)$ in dim $C(A)$?



rang A = rang B = r

$m-r$

$A\vec{x}=\vec{0}$ $\qquad$ $B\vec{x}=\vec{0}$

dim $N(A)$ = dim $N(B)$ =
= # prostih spremenljivk

r $\qquad$ št. prostih spr.

$$\boxed{\text{rang } A + \dim N(A) = n} \leftarrow \text{št. stolpcev mtr. } A$$

$$\dim N(A) = n - r$$

Št. lin. neodvisnih vrstic v A = št. neničelnih vrstic v B
= r = rang A
= št. pivotov v B
= št. lin. neodvisnih stolpcev v A

$$\boxed{\begin{aligned}\text{rang} &= \text{št. lin. neodv. vrstic} \\ &= \text{št. lin. neodv. stolpcev}\end{aligned}}$$

$\Rightarrow$ $\boxed{\dim C(A) = \text{rang } A}$ $\quad$ in $\quad$ $\boxed{\dim C(A) + \dim N(A) = n}$