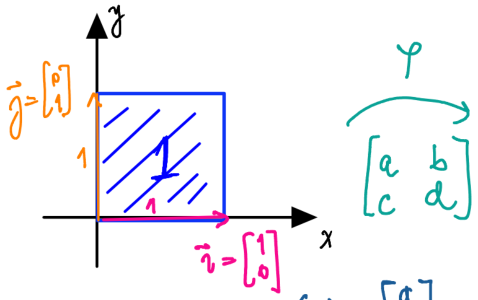
4. DETERMINANTE

Primer: $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix} \in \mathbb{R}^{2\times 2}$

Oglejmo si linearno preslikavo $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$

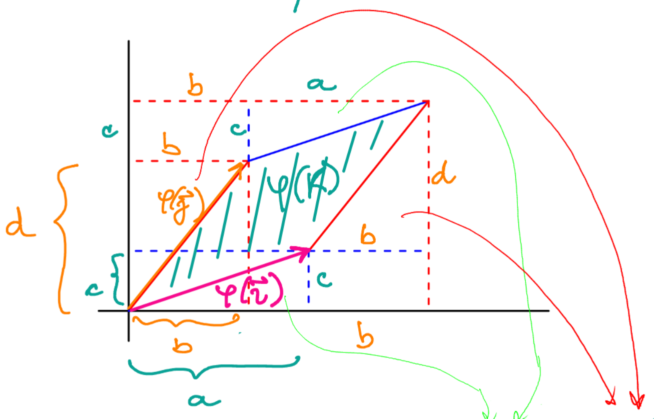
$\vec{x} \mapsto A\vec{x}$.

Naj bo $K=[0,1]\times[0,1]$ enotski kvadrat v $\mathbb{R}^2$. Zanima nas ploščina $\varphi(K)$, torej slike kvadrata K s preslikavo φ.



$\varphi(\vec{i}) = \begin{bmatrix}a\\c\end{bmatrix}$

$\varphi(\vec{j}) = \begin{bmatrix}b\\d\end{bmatrix}$



ploščina $\varphi(K) = \left|(a+b)(c+d) - 2\cancel{\frac{ac}{2}} - 2\cancel{\frac{bd}{2}} - 2bc\right| =$

$= |bc + ad - 2bc| =$

$= |ad - bc|$

Preslikavo $D: \overset{A}{\mathbb{R}^{2\times 2}} \to \mathbb{R}$, ki predstavlja predznačeno ploščino paralelograma, napetega na stolpce matrike A bomo kmalu imenovali determinanta.

Lastnosti D:

1) $D(I) = D\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 1$

2) $D(A^{(2)}, A^{(1)}) = D\begin{bmatrix} b & a\\ d & c\end{bmatrix} = bc - ad = -(ad-bc) = -D(A^{(1)}, A^{(2)})$

$A = (A^{(1)}, A^{2})$ (antisimetričnost)

3) $D(\alpha A^{(1)}, A^{(2)}) = \alpha\, D\begin{bmatrix} a & b\\ c & d\end{bmatrix} = \alpha\, D(A^{(1)}, A^{(2)})$
(homogenost v 1. stolpcu)

$D(A_1^{(1)} + A_2^{(1)}, A^{(2)}) = D(A_1^{(1)}, A^{(2)}) + D(A_2^{(1)}, A^{(2)})$
(aditivnost v 1. stolpcu)

$D\begin{bmatrix} a_1+a_2 & b\\ c_1+c_2 & d\end{bmatrix} = (a_1+a_2)d - b(c_1+c_2) =$

$= a_1d - bc_1 + a_2d - bc_2$

$= D\begin{bmatrix} a_1 & b\\ c_1 & d\end{bmatrix} + D\begin{bmatrix} a_2 & b\\ c_2 & d\end{bmatrix}$

linearnost v 1. stolpcu

(2) + 3) = linearnost v vsakem stolpcu)

Definicija: Naj bo $D : \mathbb{R}^{n\times n} \to \mathbb{R}$ funkcija z lastnostmi:

(D1) $D(I) = 1$

(D2) $D(A^{(1)},\dots,A^{(i-1)},A^{(j)},A^{(i+1)},\dots,A^{(j-1)},A^{(i)},A^{(j+1)},\dots,A^{(n)}) = -D(A^{(1)},\dots,A^{(n)})$

(D3) $D(A^{(1)},\dots,A^{(i-1)},\alpha A^{(i)}+\beta B^{(i)},A^{(i+1)},\dots,A^{(n)}) = \alpha D(A^{(1)},\dots,A^{(n)}) + \beta D(A^{(1)},\dots,A^{(i-1)},B^{(i)},A^{(i+1)},\dots,A^{(n)})$

kjer $A = [A^{(1)}, A^{(2)},\dots,A^{(n)}] \in \mathbb{R}^{n\times n}$.

Vprašanji:

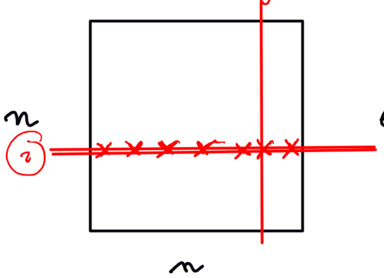
1) Ali takšna D sploh obstaja?

$n=2$ DA. $D\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = ad-bc$ ✓

$n=1$ DA. $D(a)=a$ ✓

kaj pa za $n\geq 3$?

Denimo, da $D : \mathbb{R}^{(n-1)\times(n-1)} \to \mathbb{R}$ obstaja.

$A =$  $\in \mathbb{R}^{n\times n}$, naj $A(i,j)$ matriko, ki jo dobimo iz A tako, da izbrišemo i-to vrstico in j-ti stolpec

$A(i,j) \in \mathbb{R}^{(n-1)\times(n-1)}$

za $A\in\mathbb{R}^{n\times n}$ definiramo

$$D_i(A) = (-1)^{i+1} a_{i1}\, D(\underbrace{A(i,1)}_{\in\mathbb{R}^{(n-1)\times(n-1)}}) + (-1)^{i+2} a_{i2}\, D(A(i,2)) + \dots + (-1)^{i+n} a_{in}\, D(A(i,n))$$

(RD) $$\boxed{D_i(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\, D(A(i,j))}$$ za $i=1,\dots,n$

DN: Preverite, da velja (D1)–(D3) za tako definirane D_i

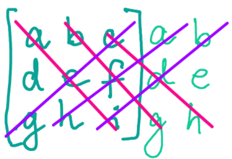
Primer: $n=1,2$ ✓

$$D_1\begin{array}{c}\begin{array}{ccc}1&2&3\end{array}\\ \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\end{array} = (-1)^{1+1} a\, D\left(\begin{bmatrix}e&f\\h&i\end{bmatrix}\right) + (-1)^{1+2} b\, D\left(\begin{bmatrix}d&f\\g&i\end{bmatrix}\right) + (-1)^{1+3} c\, D\left(\begin{bmatrix}d&e\\g&h\end{bmatrix}\right) =$$

$= a(ei-fh) - b(di-fg) + c(dh-eg) =$

$= aei - afh - bdi + bfg + cdh - ceg =$

$= \underline{aei} + \underline{bfg} + \underline{cdh} - (\underline{afh} + \underline{bdi} + \underline{ceg})$



2) Koliko je takšnih funkcij D? <u>Ena sama!</u>

Def: Takšno (edino) funkcijo $D: \mathbb{R}^{n\times n} \to \mathbb{R}$ imenujemo **determinanta** in jo označimo z

$$\det(A) = |A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

Primer: ① $\det[a] = a$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

② Naj $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}$ ← zg. Δ mtr. Koliko je $\det A$?

$$\det A \underset{\substack{(RD)\\ i=n}}{=} \sum_{j=1}^{n} (-1)^{n+j} \underbrace{a_{nj}}_{=0 \text{ razen za } j=n} \det(A(n,j)) = \underbrace{(-1)^{2n}}_{=1} a_{nn} \det A(n,n) =$$

$$= a_{nn} \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ 0 & a_{22} & \cdots & a_{2,n-1} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1,n-1} \end{bmatrix} = \text{ ← podobno nadaljujemo}$$

$$= a_{nn}\, a_{n-1,n-1}\, a_{n-2,n-2} \cdots a_{22} \underbrace{\det[a_{11}]}_{a_{11}} =$$

$$= a_{11} a_{22} a_{33} \cdots a_{nn}$$

⇒ determinanta zgornje trikotne matrike je enaka produktu diagonalnih elementov.

Lastnosti determinante ($A \in \mathbb{R}^{n\times n}$):

(1) $\det(A^{(1)}, \ldots, A^{(i-1)}, \underset{\substack{\| \\ 0\cdot A^{(i)}}}{\vec{0}}, A^{(i+1)}, \ldots, A^{(n)}) = 0 \cdot \det(A^{(1)}, \ldots, A^{(n)}) = 0$

$$\begin{array}{c} \\ (2) \end{array}\ \det\Big(A^{(1)},\dots,A^{(i-1)},\overset{\overset{i}{\downarrow}}{A^{(i)}},A^{(i+1)},\dots,A^{(j-1)},\overset{\overset{j}{\downarrow}}{A^{(i)}},A^{(j+1)},\dots,A^{(n)}\Big) =$$

zamenjamo i-ti in j-ti stolpec

$$= -\det\Big(A^{(1)},\dots,A^{(i-1)},A^{(i)},A^{(i+1)},\dots,A^{(j-1)},A^{(i)},A^{(j+1)},\dots,A^{(n)}\Big)$$

$$d=-d \Rightarrow 2d=0 \Rightarrow d=0 \Rightarrow$$

$\Rightarrow$ determinanta matrike, ki ima 2 enaka stolpca, je enaka 0.

$$(3)\ \det\Big(A^{(1)},\dots,A^{(i-1)},A^{(i)}+\alpha A^{(j)},A^{(i+1)},\dots,A^{(n)}\Big) \overset{3.}{=}$$

$$= \det\big(A^{(1)},\dots,A^{(n)}\big) + \alpha \underbrace{\det\Big(A^{(1)},\dots,A^{(i-1)},\overset{\overset{i}{\downarrow}}{A^{(j)}},A^{(i+1)},\dots,\overset{\overset{j}{\downarrow}}{}\dots,A^{(n)}\Big)}_{\text{dva enaka stolpca}} =$$

$$= \det A + \alpha\cdot 0 = \det A$$

$\Rightarrow$ determinanta matrike se ne spremeni, če nekemu stolpcu prištejemo večkratnik nekega drugega stolpca

Primer: Izračunajmo $\det\begin{bmatrix}1&2&3\\0&2&2\\0&1&3\end{bmatrix}$.

$$\begin{vmatrix}1&2&3\\0&2&2\\0&1&3\end{vmatrix} \underset{\substack{S_2\leftrightarrow S_3\\(D2)}}{=} -\begin{vmatrix}1&3&2\\0&2&2\\0&3&1\end{vmatrix} \underset{\substack{S_2\leftarrow S_2+(-3)S_3\\(D3)}}{=} -\begin{vmatrix}1&-3&2\\0&-4&2\\0&0&1\end{vmatrix} = -1\cdot(-4)\cdot 1 = 4$$

Še več uporabnih lastnosti:

$$(4)\ A,B\in\mathbb{R}^{n\times n}:\ \det(AB) = \overbrace{\det A}^{\in\mathbb{R}}\cdot\overbrace{\det B}^{\in\mathbb{R}} = \det(BA)$$

(5) Če so stolpci matrike $A\in\mathbb{R}^{n\times n}$ linearno odvisni, je $\det A = 0$.

Kaj pa obratno? Če je $\det A = 0 \Rightarrow$ stolpci A so lin. odvisni.

(Zakaj? Če stolpci A lin. neodv. $\Rightarrow A$ obrnljiva

$$\Rightarrow A\cdot A^{-1} = I$$
$$\Rightarrow \det(AA^{-1}) = \det I$$
$$\overset{(4)}{\Rightarrow} (\det A\cdot \det A^{-1}) = 1 \qquad (*)$$
$$\Rightarrow \det A \neq 0\ )$$

$A\in\mathbb{R}^{n\times n}$ $\boxed{A \text{ obrnljiva} \Leftrightarrow \det A\neq 0}$

(6) $A \in \mathbb{R}^{n \times n}$ obrnljiva $\overset{(*)}{\Rightarrow}$ $(\det A)(\det A^{-1}) = 1 \quad / : \det A \neq 0$

$$\Rightarrow \boxed{\det A^{-1} = \frac{1}{\det A}}$$

(7) $A \in \mathbb{R}^{n \times n}$: $\boxed{\det A^T = \det A}$ (verjamemo, ne dokazujemo)

$\Rightarrow$ Pri vseh lastnostih (D1) – (D3), (1), (2), (3) lahko "stolpec" zamenjamo za "vrstica".

Prepišite si jih.

Kako računamo determinanto?

RECEPT 1: z "negaussovo eliminacijo":

- za vsako menjavo vrstic/stolpcev spremenimo predznak determinante,
- večkratnik vrstice/stolpca lahko izpostavimo,
- vsaki vrstici/stolpcu lahko prištejemo večkratnik druge vrstice/stolpca.

CILJ: zgornje/spodnje trikotna matrika (katere determinanta je produkt diagonalnih elementov)

Primer: izračunajmo

$$\det \begin{vmatrix} 0 & 2 & -1 & -2 \\ 0 & 4 & 4 & 1 \\ 2 & 0 & -1 & 4 \\ -1 & 2 & 0 & 0 \end{vmatrix} \underset{v_1 \leftrightarrow v_4}{=} - \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 4 & 4 & 1 \\ \cancel{2}\ 0 & -1 & 4 \\ 0 & 2 & -1 & -2 \end{vmatrix} \underset{v_3 \leftarrow v_3 + 2v_1}{=} - \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 4 & 4 & 1 \\ 0 & 4 & -1 & 4 \\ 0 & 2 & -1 & -2 \end{vmatrix} \underset{v_2 \leftrightarrow v_4}{=}$$

$$= \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & -2 \\ 0 & 4 & -1 & 4 \\ 0 & 4 & 4 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 6 & 5 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & -43 \end{vmatrix} = (-1) \cdot 2 \cdot 1 \cdot (-43) = \underline{\underline{86}}$$

$v_3 \leftarrow v_3 - 2v_2$
$v_4 \leftarrow v_3 - 2v_2$

$v_4 \leftarrow v_4 - 6v_3$

RECEPT 2: z razvojem po vrstici i:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A(i,j)$$

ali z razvojem po stolpcu i:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ji} \det A(j,i)$$

ta dva elementa sta 0 !!

Primer: izračunajmo $\det \begin{bmatrix} 0 & 2 & -1 & -2 \\ 0 & 4 & 4 & 1 \\ 2 & 0 & -1 & 4 \\ -1 & 2 & 0 & 0 \end{bmatrix} =$ razvoj po 1. stolpcu

pojdigamo se na to, kaj je ta determinanta

ker se bosta pomnožili z 0, jih ne!

spet veliko ničel!!

za 3x3 determinanto imamo recept

$$= 0 \cdot \left| \; * \; \right| - 0 \cdot \left| \; * \; \right| + 2 \begin{vmatrix} 2 & -1 & -2 \\ 4 & 4 & 1 \\ 2 & 0 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 2 & -1 & -2 \\ 4 & 4 & 1 \\ 0 & -1 & 4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 4 & 4 \\ 0 & -1 \end{matrix}$$

razvoj po 3. vrstici

$- \; - \; - \; + \; + \; +$ 6 členov

$$= 2 \cdot \left(2 \cdot \begin{vmatrix} -1 & -2 \\ 4 & 1 \end{vmatrix} - 0 \left| \; * \; \right| + 0 \left| \; * \; \right| \right) + \big(2\cdot 4\cdot 4 + (-1)\cdot 1\cdot 0 + (-2)\cdot 4\cdot(-1) -$$

$$- (-2)\cdot 4\cdot 0 - 2\cdot 1\cdot(-1) - (-1)\cdot 4\cdot 4\big) =$$

$$= 4(-1+8) + (32 + 8 + 2 + 16) =$$

$$= 28 + 58 = \underline{\underline{86}}$$

(8) Računanje inverzov matrik

$A \in \mathbb{R}^{n\times n}$, $A(i,j) \in \mathbb{R}^{(n-1)\times(n-1)}$ podmatrika A brez i-te vrstice in j-tega stolpca

$$k_{ij} = (-1)^{i+j} \det(A(i,j))$$ kofaktor matrike

Naj $K_A = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}^T \in \mathbb{R}^{n\times n}$ ← prirejenka matrike A. Izračunajmo AK_A.

• diag. eleni:

$$(AK_A)_{ii} = \sum_{j=1}^{n} A_{ij} (K_A)_{ji} = \sum_{j=1}^{n} A_{ij} k_{ij} = \sum_{j=1}^{n} A_{ij} (-1)^{i+j} \det A(i,j) \overset{\text{razvoj det } A \text{ po } i\text{-ti vrstici}}{=} \det A$$

• izvendiag. eleni, $i \neq j$

$$(AK_A)_{ij} = \sum_{l=1}^{n} A_{il} (K_A)_{lj} = \sum_{l=1}^{n} A_{il} k_{jl} = \sum_{l=1}^{n} A_{il} (-1)^{j+l} \det A(j,l) = 0$$

razvoj det M, M ima dve vrstici enaki

$$AK_A = \begin{bmatrix} \det A & & & 0 \\ & \det A & & \\ & & \det A & \\ 0 & & & \ddots \\ & & & & \det A \end{bmatrix} = (\det A) \cdot I_n \quad / : \det A \neq 0$$

A obrnljiva

$$A \left(\frac{1}{\det A} \cdot K_A \right) = I_n \quad \Rightarrow \quad \boxed{A^{-1} = \frac{1}{\det A} \cdot K_A}$$

Primer: $A = \begin{bmatrix} -1 & -3 & -2 \\ 0 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ Izračunajmo A^{-1}, če obstaja.

$$\det A \underset{\substack{\text{razvoj po} \\ \text{2. vrstici}}}{=} 1 \cdot \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} = -1 + 6 = 5$$

, $\det A = 5 \neq 0 \Rightarrow A$ je obrnljiva

$+\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$ $\qquad -\begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} = 0$

$+\begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} = -3$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 0 & -3 \\ 7 & 5 & -11 \\ 2 & 0 & -1 \end{bmatrix}^T =$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 7 & 2 \\ 0 & 5 & 0 \\ -3 & -11 & -1 \end{bmatrix}$$

DN: $A \cdot A^{-1} = \dots = I$