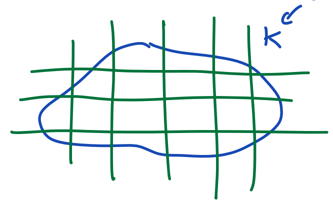
③ Dvojni integral

Ideja: Integriramo po območju v ravnini:

$K \subset \mathbb{R}^2$, $\quad f: K \to \mathbb{R}$ zvezna



za $n \in \mathbb{N}$ definiramo Riemannovo vsoto S_n:

- Ravnino razdelimo na kvadratno mrežo $\mathcal{M}_n$, katere kvadratki so dimenzij $\frac{1}{n} \times \frac{1}{n}$:

$$\mathcal{M}_n = \{ \underbrace{[\tfrac{k}{n}, \tfrac{k+1}{n}] \times [\tfrac{\ell}{n}, \tfrac{\ell+1}{n}]}_{C_{k,\ell}} \; ; \; k, \ell \in \mathbb{Z} \}$$

lahko bi vzeli točko v $C_{k,\ell}$

- $$S_n = \sum_{C_{k,\ell} \subset K} f\left(\tfrac{k}{n}, \tfrac{\ell}{n}\right) \cdot \underbrace{\text{Ploščina } C_{k,\ell}}_{\left(\frac{1}{n}\right)^2}$$

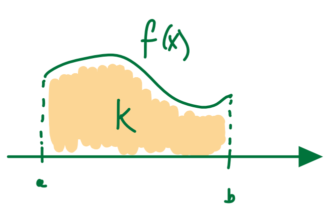
S_n' dobimo z "$C_{k,\ell} \cap K \neq 0$" $\rightsquigarrow$ $\lim\limits_{n\to\infty} S_n' = \lim\limits_{n\to\infty} S_n$

Def: $$\iint_K f(x,y)\,dx\,dy = \lim_{n\to\infty} S_n .$$ ← limita neodvisna od omenjenih izbir

Integriranje v praksi:
- K parametriziramo z x in y
- dvojni integral $\rightsquigarrow$ dvakratni integral
- Vrstni red integriranja ni pomemben (Fubinijev izrek)

Primer:
- $$\iint_{[0,1]^2} x^2\,dx\,dy = \int_0^1 \underbrace{\int_0^1 x^2\,dy}_{1}\,dx = \int_0^1 1 \cdot x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \underline{\underline{\frac{1}{3}}}$$

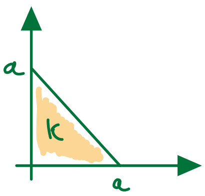
- Ploščina pod grafom f: $$\iint_K dx\,dy = \int_a^b \int_0^{f(x)} dy\,dx = \int_a^b f(x)\,dx$$



Opazka: Ploščina $K = \iint_K \underbrace{dx\,dy}$

vsota

ploščine majhnih kvadratkov v K

- 

$$\iint_K dx\,dy = \int_0^a \int_0^{a-x} dy \quad dx = \int_0^a (a-x)\,dx = ax - \frac{x^2}{2}\Big|_0^a =$$

$$= a^2 - \frac{a^2}{2} = \frac{a^2}{2}$$

ploščina trikotnika