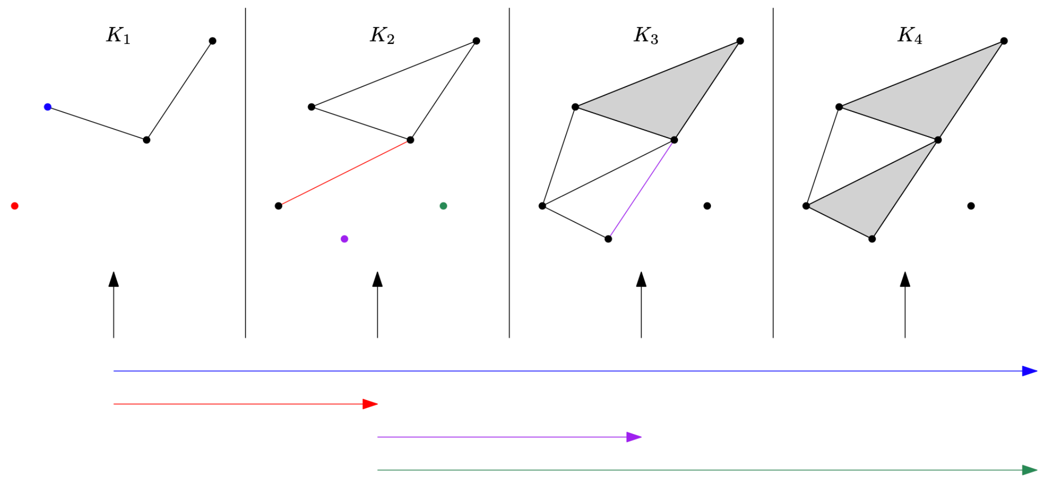
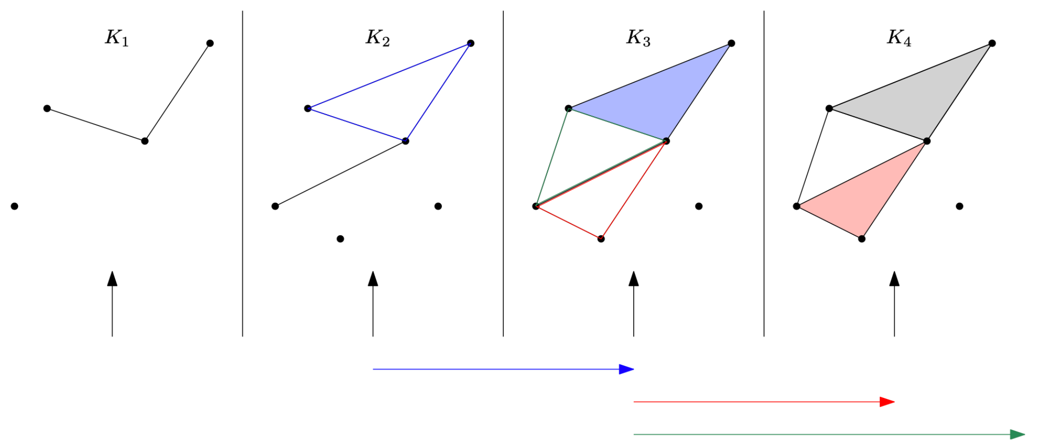
Persistent homology

① Idea

Track the evolution of holes along a growing scx.

Motivate with technique used in Brouwer proof.





② Formal definition

Def: K a scx. A discrete filtration of K is a sequence of subcomplexes:

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m = K.$$

⟸ a model of a growth

In each step we add simplices

Adding a single sx either creates or fills in a hole.

A filtration can be expressed as a sequence of inclusions

$$K_1 \overset{i_{1,2}}{\hookrightarrow} K_2 \overset{i_{2,3}}{\hookrightarrow} \cdots \hookrightarrow K_m = K$$

$i_{s,*}: K_s \longrightarrow K_*$ the obvious composition

Fix a field $\mathbb{F}$, dimension $q \in \{0,1,\dots\}$.

APPLY homology $H_q(-;\mathbb{F})$ to the filtration

$(i_{s,t})_*$ not necessarily injections

$$H_q(K_1;\mathbb{F}) \xrightarrow{(i_{1,2})_*} H_q(K_2;\mathbb{F}) \xrightarrow{(i_{2,3})_*} \cdots \longrightarrow H_q(K_m;\mathbb{F}) = H_q(K;\mathbb{F})$$

Def: Persistent homology groups: images of maps $(i_{s,t})_*$, i.e. $\{(i_{s,t})_*(H_q(K_s;\mathbb{F}))\}_{s\le t}$

Persistent Betti numbers $\beta^q_{s,t}$: ranks of persistent homology groups.

Example:

$\beta^0_{s,t} \rightarrow$

s \ t	1	2	3	4
1	2	1	1	1
2	/	3	2	2
3	/	/	2	2
4	/	/	/	2

$\beta^1_{s,t} \rightarrow$

s \ t	1	2	3	4
1	0	0	0	0
2	/	1	0	0
3	/	/	2	1
4	/	/	/	1

③ Visualisation   How to obtain barcode from persistent Betti numbers?

$n_{s,t}$ ..... # of bars from s to t. $[s,t)$

represents dim of homology born AT s and terminating AT t.

$\beta_{s,t}$ represents dim of homology born BY s and terminating after t.

Homology born at s : $H_q(K_s) / \mathrm{Im}(i_{s-1,s})_*$

Its dimension is $\beta_{s,s} - \beta_{s-1,s}$

Homology terminating at t: $\mathrm{Ker}(i_{t-1,t})_*$

Its dimension is $\beta_{t-1,t-1} - \beta_{t-1,t}$

$$\Longrightarrow \quad n_{s,t} = \underbrace{(\beta_{s,t-1} - \beta_{s-1,t-1})}_{\text{dimension of homology born @ s still alive @ t-1}} - \underbrace{(\beta_{s,t} - \beta_{s-1,t})}_{\text{dimension of homology born @ s still alive @ t}}$$

$\Longrightarrow \quad n_{s,\infty} = \beta_{s,m} - \beta_{s-1,m}$ surviving homology

Example: Extract barcodes from example above

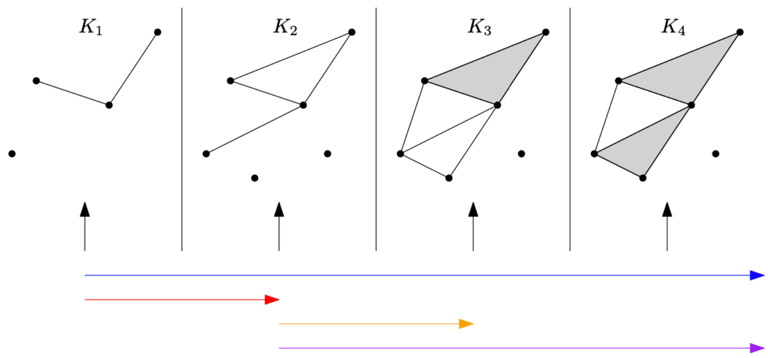
$$n_{2,3} = (1-1) - (3-2) = 1$$

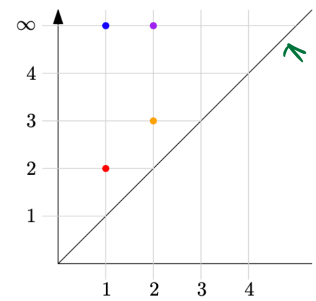
$\beta^0_{s,t} \rightarrow$

s \ t	1	2	3	4
1	2	1_-	1_+	1
2	/	3_+	2_-	2
3	/	/	2	2
4	/	/	/	2

$n_{s,t}$ can be visualized as a barcode or persistence diagram.

each pt has a multiplicity





diagonal: lifespan 0.

$\{\beta_{s,t}\}_{s \le t}$ determine $n_{s,t}$ and vice versa

Fundamental lemma of persistent homology:

$$\beta_{s,t} = \sum_{s' \le s,\, t' \ge t} n_{s',t'}.$$

indexing includes ∞

Proof: Clear from the context.

④ Computation (get $n_{s,t}$ directly)

Fix a field $\mathbb{F}$, filtration $K_1 \le K_2 \le \dots \le K_m$

Assumption: We are adding one sx at a time: $K_n = K_{n-1} \cup \{\sigma_n\}$ ← can be d one

Adding sx $\sigma_i^{(p)}$ to K results in a change of homology:

if $[\partial\sigma] \in H_{p-1}(K_{i-1})$ non-zero

Adding σ makes $[\partial\sigma] = 0$ in $H_{p-1}(\underbrace{K_{i-1} \cup \{\sigma\}}_{K_i})$

σ is a terminal sx, terminates $[\partial\sigma]$

equivalent to: $\partial\sigma \notin lin\{\partial\tau_j\}_{\tau_j^p \in K_{i-1}}$

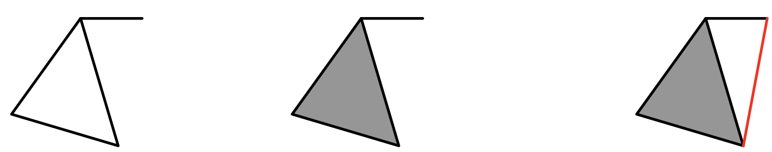
if $[\partial\sigma] \in H_{p-1}(K_{i-1})$ is ZERO

equivalent to: $\partial\sigma = \sum_{\tau_j^p \in K_{i-1}^d} \lambda_j \partial\tau_j$

Adding σ CREATES $[\sigma - \Sigma\tau_i] \overset{\neq 0}{\in} H_p(\underbrace{K_{i-1} \cup \{\sigma\}}_{K_i})$

σ is a BIRTH sx, creates homology

Example:



Matrix reduction

① Use the order of sx's given by filtration to assemble boundary matrix M_g.

② Reduce matrix left-to-right using column reduction. For each column repeat:

ⓐ Determine pivot

ⓑ Subtract a previous column with the same pivot if existent, else halt

③ Extract persistence:

ⓐ pivots determine birth sx - terminal sx pairs ⇝ finite bar

(birth sx ↙ row of pivot; terminal sx ↓ column of sx.)

a representative → Representative: the column containing the pivot

ⓑ non-paired sxes are called essential sxes ⇝ infinite bar ← they are all birth sxes

they give birth to ever-lasting homology.

OBSERVATION:

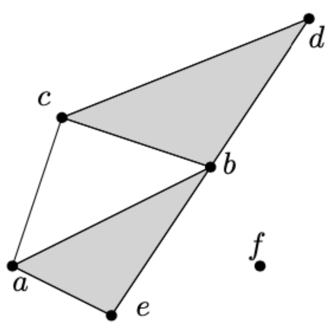
birth sxes:	terminal sxes
→ their column reduces to 0	→ their columns do not reduce to 0
→ a birth sx is non-essential iff it appears in a pivot row	→ they do not appear in a pivot row
	⇝ shortcut called TWIST

Representative of essential sx σ:

$\sigma - \sum_i \tau_i$ where $\partial\sigma - \sum_i \partial\tau_i = 0$ is the reduction of σ-column.

not a good representative for finite bars

Example: Computation for filtration given above.



$M_2 = M_2' =$

	$\langle b,c,d\rangle$	$\langle a,b,e\rangle$
$\langle b,c\rangle$	1	
$\langle b,d\rangle$	−1	
$\langle a,b\rangle$		1
$\langle c,d\rangle$	1	
$\langle a,c\rangle$		
$\langle a,e\rangle$		−1
$\langle b,e\rangle$		1

$\langle a,c\rangle$ unpaired → essential sx

We now perform the labelled matrix reduction as described above.

$M_1 =$

	$\langle b,c\rangle$	$\langle b,d\rangle$	$\langle a,b\rangle$	$\langle c,d\rangle$	$\langle a,c\rangle$	$\langle a,e\rangle$	$\langle b,e\rangle$
$\langle a\rangle$			−1		−1	−1	
$\langle b\rangle$	−1	−1	1				−1
$\langle c\rangle$	1			−1	1		
$\langle d\rangle$		1		1			
$\langle e\rangle$						1	1
$\langle f\rangle$							

	$\langle b,c\rangle$	$\langle b,d\rangle$	$\langle a,b\rangle$	$\langle c,d\rangle$	$\langle a,c\rangle$	$\langle a,e\rangle$	$\langle b,e\rangle$
$\langle a\rangle$			−1		−1	−1	
$\langle b\rangle$	−1	−1	1				−1
$\langle c\rangle$	1				1		
$\langle d\rangle$		1					
$\langle e\rangle$						1	1
$\langle f\rangle$							

	$\langle b,c\rangle$	$\langle b,d\rangle$	$\langle a,b\rangle$	$\langle c,d\rangle$	$\langle a,c\rangle$	$\langle a,e\rangle$	$\langle b,e\rangle$
$\langle a\rangle$			−1			−1	
$\langle b\rangle$	−1	−1	1				−1
$\langle c\rangle$	1						
$\langle d\rangle$		1					
$\langle e\rangle$						1	1
$\langle f\rangle$							

pair (e, ae)

$M_1' =$

	$\langle b,c\rangle$	$\langle b,d\rangle$	$\langle a,b\rangle$	$\langle c,d\rangle$	$\langle a,c\rangle$	$\langle a,e\rangle$	$\langle b,e\rangle$
$\langle a\rangle$			−1			−1	
$\langle b\rangle$	−1	−1	1				
$\langle c\rangle$	1						
$\langle d\rangle$		1					
$\langle e\rangle$						1	
$\langle f\rangle$							

$\langle a\rangle, \langle c\rangle$ essential components

bar born with $\langle e\rangle$ terminates with $\langle ae\rangle$