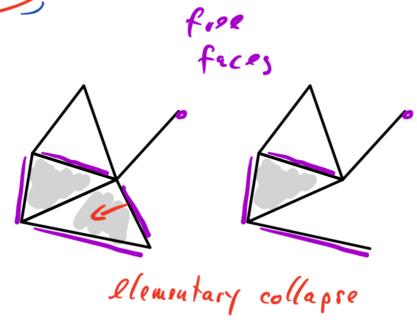
Discrete Morse Theory

① Motivation (An easy simplification of a scx)

Def: • K scx. $\sigma \in K$ is a free face if it is a face of exactly one sx $\tau \in K$.

τ is a max face, $\dim \tau = \dim \sigma + 1$

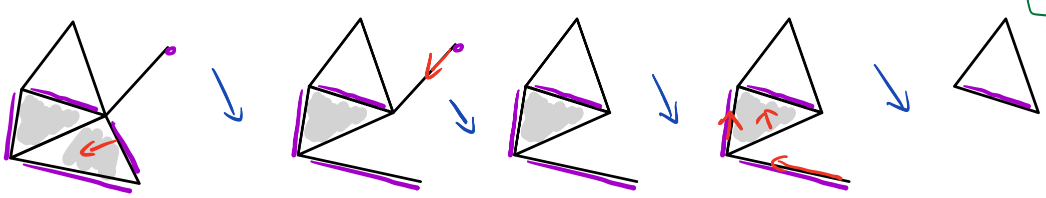


• An elementary collapse is a removal $K \rightsquigarrow K \setminus \{\sigma, \tau\}$. In this case $K \setminus \{\sigma,\tau\} \hookrightarrow K$ is a homotopy equivalence.

• Scx K is collapsible [$K \searrow L$] to a sub scx $L \subseteq K$ if there exists a collapse (i.e., a sequence of elementary collapses) reducing K to L.

• K is collapsible if $K \searrow \cdot$.

$\| \quad K \searrow L \;\; \underset{\Longrightarrow}{\not\Leftarrow} \;\; K \simeq L$ → Dunce hat, Bing's house



IDEA: let's try to formalize collapsing sequence.

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② Discrete Morse functions [DMF] and discrete vector fields [DVF]

Def: K scx. A function $f: K \to \mathbb{R}$ is a DMF if $\forall \sigma^k \in K$:

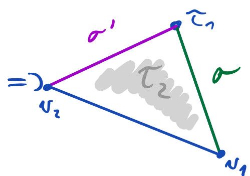
ⓐ $e_1 = |\{\tau^{k-1} < \sigma \; ; \; f(\tau) \geq f(\sigma)\}| \leq 1$, and

ⓑ $e_2 = |\{\tau^{k+1} > \sigma \; ; \; f(\tau) \leq f(\sigma)\}| \leq 1$.

$\underbrace{\left.\begin{array}{l}\text{faces}\\ \text{cofaces}\end{array}\right\}\text{ where } f \text{ does not abide by dim.}}_{\text{exceptional face or coface}}$

Prop: $e_1 \cdot e_2 = 0$.

Proof: Assume $\tau_1^{k-1} \leq \sigma$ is exceptional for σ, $\tau_2^{k+1} \geq \sigma$ is exceptional for σ $\Rightarrow$



$f(\tau_1) > f(\sigma) > f(\tau_2)$

$f(\tau_1) < f(\sigma') < f(\tau_2)$

$\rightarrow\leftarrow$

Cor: σ is an except. face of τ iff τ is an except. coface of σ.

Pairs (sx, except. face) are disjoint.

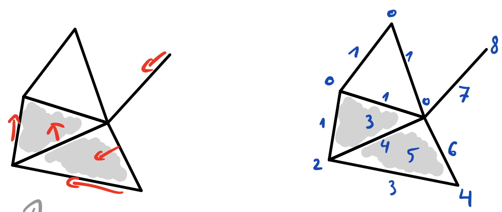
Def: Given a DMF f on a sx K, a pair (sx σ, except. face of σ) is called a **regular pair**.

Sxes of K are partitioned into: → regular pairs (indicated by arrows).

→ critical sxes (where f completely respects dim.)

(fewer crit. sxes, better simplification)

Example: Collapse is induced by a DMF (not uniquely).



let's formalize these arrows

Prop: n_i .... # of critical n-sxes

Then $\chi = n_0 - n_1 + n_2 - \dots$

Proof: removing a regular pair preserves χ. ∎

Def: K sx. A **discrete vector field** [DVF] on K is a disjoint collection of pairs $\underbrace{(\sigma_i, \tau_i)}$ of sxes of K, such that σ_i is a face of τ_i, $\forall i$. Critical sxes NOT INVOLVED.

↳ such pairs are called arrows.

A DVF is called a **discrete gradient vector field** [DGVF] if it is induced by a DMF (as a collection of regular pairs).

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③ DGVF's (recognizing DGVF's)

Def: K sx, $p \in \mathbb{N}$. Given a DVF on K consisting of pairs $\{(\sigma_j, \tau_j)\}_{j \in J}$, a **$p$-path** is a sequence

sup sx

$$\underbrace{\sigma_{j_1}^{p-1} \to \tau_{j_1}^{p}} \geq \underbrace{\sigma_{j_2}^{p-1} \to \tau_{j_2}^{p}} \geq \sigma_{j_3}^{p-1} \to \dots \to \tau_{j_k}^{p} \geq \sigma_{k+1}^{p-1}$$

arrows in a DVF

Such a path is a **cycle** if $\sigma_1 = \sigma_{k+1}$ and $k \geq 1$.

A DVF is **acyclic** if it admits no cycle.

Observations: ⓐ a crit. sx can only appear as the last sx in a p-path

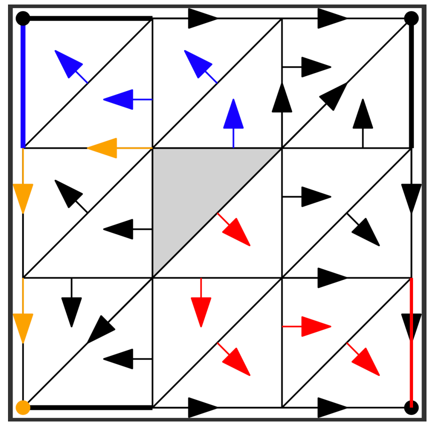
ⓑ Given a DMF f, function values decrease along any p-path.

$$f(\sigma_{j_i}) \geq f(\tau_{j_i}) > f(\sigma_{j_{i+1}}), \ \forall i.$$

In particular: $f(\sigma_{i_1}) > f(\sigma_{i_m})$, $\forall m > 1$.

Ⓒ Ⓑ implies each DGVF is acyclic.

↙ converse



Examples of paths

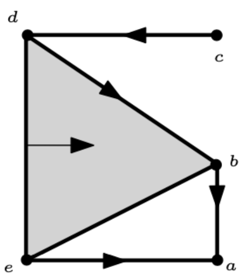
THM: Each acyclic DVF on K is a GDVF.

(Related to: a vector field on $\mathbb{R}^2$ with zero curl is a grad. field)

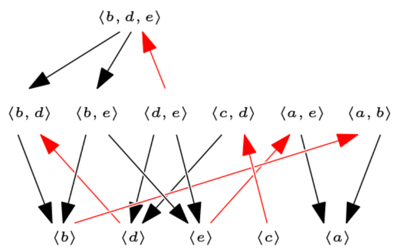
Proof by example:

Acyclic DVF

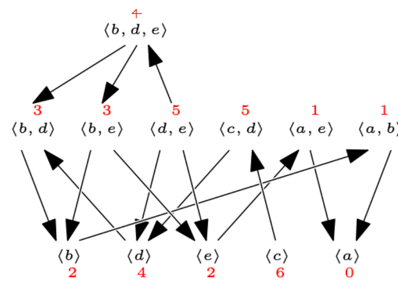
↙



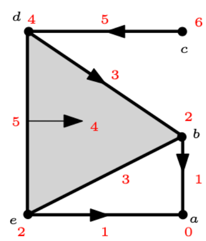
Modified Hasse diagram encoding conditions on f.



acyclic graph



reverse engineer values of f.



∎

Prop: Suppose critical sxes of an acyclic DVF on K form a subcx $L \leq K$. Then $K \searrow L$ and thus $K \simeq L$.

Proof: Claim: $\exists$ a regular pair (σ, τ) with σ being a free face.

↓ justification

Let n — max dim of a sx in $K \setminus L$.
Choose a maximal n-path.
Let (σ, τ) be its initial pair

σ is a free face:

- $\sigma < \tau$ by def
- If σ was a facet of a sx in $K \setminus L$, the pair of that sx could be used to prolong our p-path. $\rightarrow\leftarrow$
- If σ was a facet of a sx in L, σ itself would be in L $\rightarrow\leftarrow$ □

Cor: If an acyclic DVF on K has a single critical sx, then $K \simeq \bullet$.

how to use it

- Remove the pair (σ, τ) by an elementary collapse
- Inductively use the claim to proceed
- end @ L.

∎

④ Morse homology (How to compute homology from critical sxes).

discrete ← setting

Classical Morse theory: Obtain homology or a homotopy type of a manifold from critical points of a function on it. (on S^1: # of MAX = # of MIN)

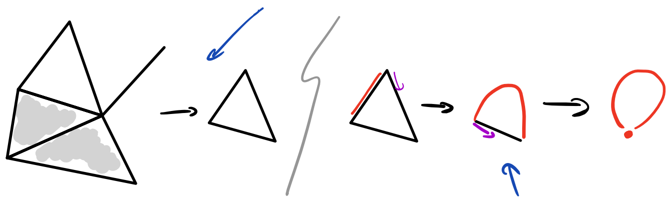
Similarly: a manifold admits a non-trivial tangent vector field $\Rightarrow \chi = 0$.

(maybe sketch it)

[Simple to prove for DVF's]

(show it!)

IDEA: last time we simplified.



Today we choose a crit sx and continue to further simplification (not a sx).

SETTING: K a scx with a gradient vector field, G a group for coefficients

n_i ... # of critical i-sxes

Def: let $p \in \{0,1,..\}$. A Morse p-chain is a formal sum

$$\sum_{i=1}^{n_p} \lambda_i \cdot \sigma_i^p \qquad \lambda_i \in G, \quad \sigma_i^p \in K \text{ oriented critical } p\text{-sx}$$

Morse chain group C_p is the group of Morse p-chains (with the obvious operations).

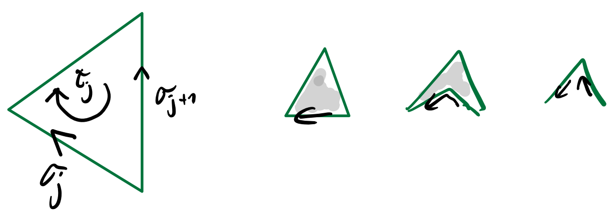
An oriented p-path from an oriented sx σ_1^{p-1} to an oriented sx σ_{k+1}^{p-1} is a p-path

$$\sigma_1^{p-1} \to \tau_1^p \geq \sigma_2^{p-1} \to \tau_2^p \geq \sigma_3^{p-1} \to \dots \to \tau_k^p \geq \sigma_{k+1}^{p-1}$$

consisting of oriented sxes, such that for each j the orientation induced by τ_j on its faces:

① Matches σ_j

② Does not match σ_{j+1}



Given oriented sx τ^p let $\delta(\tau)$ denote the collection of all of its facets with the induced orientation arising from τ.

For each oriented critical $(p-1)$-sx σ let

$$\alpha_{\tau,\sigma} = \sum_{\sigma' \in \delta(\tau)}' \left| \{\text{oriented } p\text{-paths from } \sigma' \text{ to } \sigma\} \right|$$

Def: The boundary map ∂ on C_p is defined as follows (on the generators $\in$ crit. p-sxs)

$$\partial_p \tau = \sum_{i=1}^{n_{p-1}} \Big(\underbrace{\alpha_{\tau,\sigma_i}} - \underbrace{\alpha_{\tau,-\sigma_i}} \Big) \cdot \sigma_i$$

in general not related

Morse chain complex:

$$\cdots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

It turns out $\partial^2 = 0$.

Morse homology

$$\mathcal{H}_p(K;G) = {}^{\ker \partial_p}\!/\!_{\operatorname{Im} \partial_{p+1}}$$

quotient also depends on grad. vect. field, but not the homology

THM: Morse homology is isomorphic to simplicial homology:

$$\mathcal{H}_p(K;\mathbb{F}) \cong H_p(K,\mathbb{F}).$$

Corollary: $\forall p \quad n_p \geq b_p$ ← Betti number

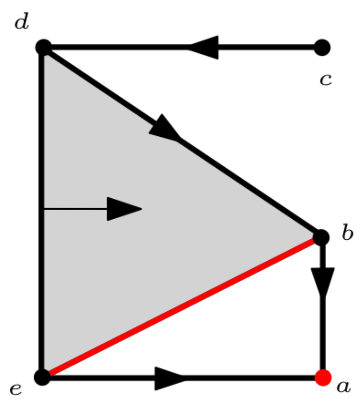
# of crit p-sxes

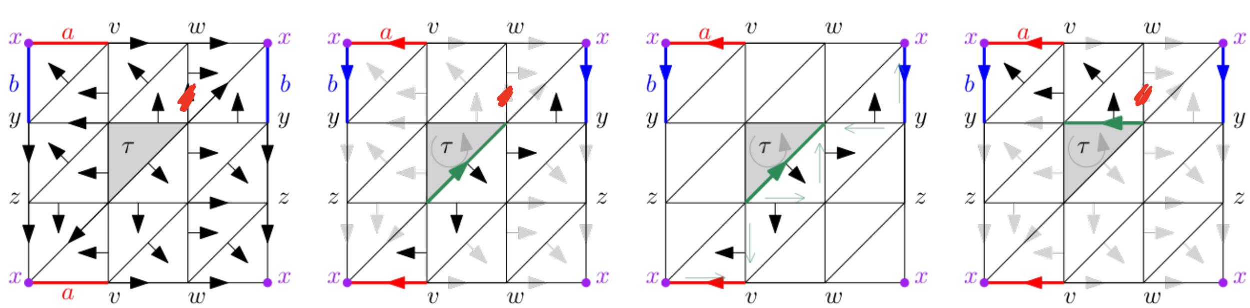
⟶ If for some DMF we have $n_p = b_p, \forall p$, f is called perfect (and $\partial_p = 0, \forall p$ if G is a field).

We usually strive to get it but not all spaces admit it. [Example: Dunce hat $\simeq$ but no free face].

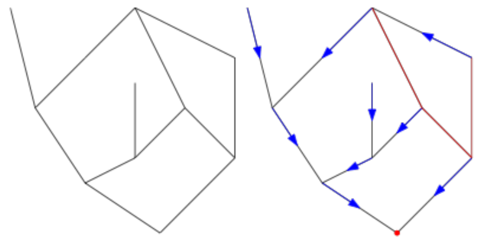
Examples:

generate DMF first





- generating DMF on graphs $\xrightarrow{\text{via spanning tree}}$
- @ Čech small encl. rad. is a gen. DMF.
- Idea of proof of Main thm:
  what does an arrow do to boundary matrices



a) (σ^{p-1}, τ^{p}) an arrow, σ a free face:

i) $\partial\sigma$ is lin dep. on other columns in ∂_p as

$$\partial\sigma = \partial\partial\tau \setminus \partial\sigma = \sum_{\substack{\sigma' \text{ facet of } \tau \\ \sigma' \neq \sigma}} \partial\sigma'$$

removing σ decreases dim $\ker \partial_p$ by 1.

ii) $\partial\tau$ is NOT lin dep. on other columns as it is the only column with σ

removing τ decreases dim $\operatorname{Im} \partial_{p+1}$

As a result dim H_p is preserved.

b) (σ^{p-1}, τ^{p}) an arrow, σ NOT a free face: similar with column reductions.