Mathematical Modelling Exam

May 27th, 2024

You have 75 minutes to solve the problems. The numbers in $[\cdot]$ represent points.

1. Answer the following questions. In YES/NO questions **verify your reasoning**.

   (a) **[1]** $f(t) = \begin{pmatrix} 2\sin t - 3 \\ 2\cos t + 4 \end{pmatrix}$, $t \in [0, 2\pi]$, is a circle. YES/NO

   (b) **[2]** $f(\varphi_1, \varphi_2, \varphi_3) = (\sin\varphi_2 \cos\varphi_1, \cos\varphi_2, \sin\varphi_2 \sin\varphi_1 \cos\varphi_3, \sin\varphi_2 \sin\varphi_1 \sin\varphi_3)$, $\varphi_1, \varphi_2 \in [0, \pi]$, $\varphi_3 \in [0, 2\pi]$ is a parametrization of a sphere in $\mathbb{R}^4$. YES/NO

   (c) **[1]** There exists an analytic solution to the differential equation
   $$y'(x) = (x^3 + 3\sin x + x^2)e^{y^2}.$$
   YES/NO

   (d) **[1]** The translation of the second order ODE $x'' - 4x' + x = 0$ into first order system of ODEs is
   $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
   YES/NO

   (e) **[1]** Let
   $$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2, \\ \dot{x}_2 &= 2x_1 - 6x_2 \end{aligned}$$
   by a system of differential equations. Then $\lim_{t\to\infty} x_1(t) = 0$ independently of the initial conditions $x_1(0)$, $x_2(0)$. YES/NO

2. (a) **[2]** Sketch the graphs of the functions
   $$f(x) = 2x + \cos(x) \quad \text{and} \quad g(x) = 2x + \sin(x)$$
   for $x \in [0, 2\pi]$. Determine the local extrema of f, g on $[0, 2\pi]$. (You do not need to determine regions of convexity/concavity.)

   (b) **[3]** Sketch the closed curves given in polar coordinates by
   $$r_1(\varphi) = 2\varphi + \cos\varphi \quad \text{and} \quad r_2(\varphi) = 2\varphi + \sin\varphi.$$

   (c) **[5]** Compute the area of the bounded region determined by the curves on the interval $\varphi \in [0, 2\pi]$. **Hint:** $\cos^2\varphi = \frac{1+\cos(2\varphi)}{2}$.

3. Let
   $$y' = -2xy + e^{-x^2+2x}, \quad y(0) = 1$$
   be the DE.

   (a) **[4]** Solve the DE explicilty.

   (b) **[4]** Use Runge–Kutta method with the Butcher tableau
   $$\begin{array}{c|ccc} 0 & 0 & & \\ \frac{1}{2} & \frac{1}{2} & 0 & \\ 1 & -1 & 2 & 0 \\ \hline & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{array},$$
   and the step-size $h = 0.1$ to compute the approximation $y_1 \approx y(0.1)$.

   (c) **[1]** Estimate the error of the numerical solution of $y(0.1)$.