Computational topology

Lab work, 7^{th} week

1. Write the addition table for

   (a) $\mathbb{Z}_2 = \langle a \mid 2a = 0 \rangle$,

   (b) $\mathbb{Z}_3 = \langle a \mid 3a = 0 \rangle$,

   (c) $\mathbb{Z}_4 = \langle a \mid 4a = 0 \rangle$,

   (d) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a, b \mid 2a = 0,\ 2b = 0 \rangle$,

   (e) $\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \langle a, b \mid 2a = 0,\ 3b = 0 \rangle$.

2. The simplicial complexes X and Y are given as lists of simplices:

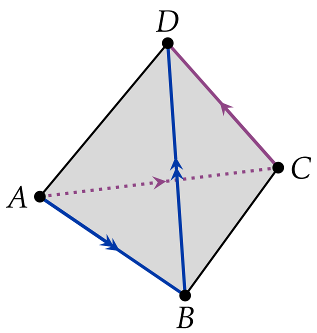
$$\begin{aligned} X &= \{A, B, C, AB, AC, BC\}, \\ Y &= \{A, B, C, D, AB, AD, BC, CD\}. \end{aligned}$$

   (a) Construct the cones CX and CY by listing all the simplices.

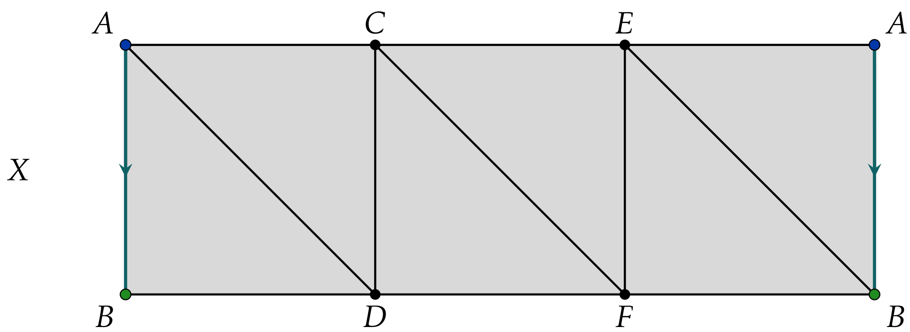
   (b) Find the sequences of collapses that simplify CX and CY as much as possible.

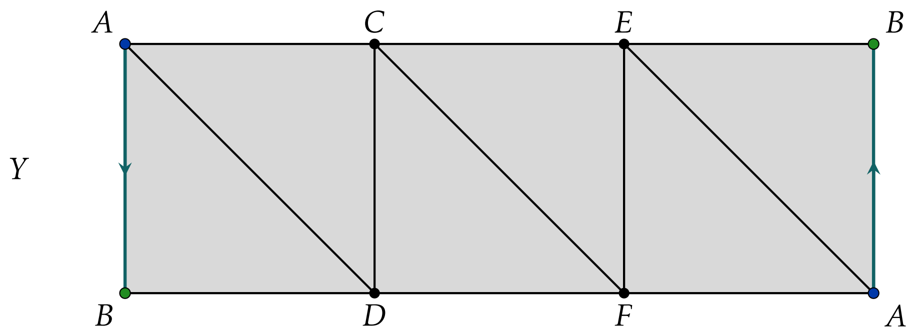
   (c) Is CX a collapsible complex for all X?

3. Let $X = \Delta^3$ be the standard 3-simplex (tetrahedron) with vertices A, B, C and D. We obtain Y by identifying the edges AB and BD and the edges AC and CD (preserving the ordering of vertices). Show that Y collapses onto a Klein bottle.



4. Given the following triangulations of the cylinder X and the Moebius band Y, find a sequence of elementary collapses that simplifies them as much as possible, then compute the homology groups $H_*(X)$ and $H_*(Y)$.





5. For the simplicial complex X in the figure below

   (a) write down the chain groups $\mathcal{C}_n$,

   (b) determine the boundary homomorphisms $\partial_n \colon \mathcal{C}_n \to \mathcal{C}_{n-1}$,

   (c) find the cycles $Z_n = \ker \partial_n$,

   (d) find the boundaries $B_n = \mathrm{im}\, \partial_n$,

   (e) determine the simplicial homology groups with $\mathbb{Z}$ coefficients, $H_n(X;\mathbb{Z})$,

   (f) determine the simplicial homology groups with $\mathbb{Z}_2$ coefficients, $H_n(X;\mathbb{Z}_2)$,

   (g) determine the Betti numbers of X and

   (h) compute the Euler characteristic of X.