Computational topology

Lab work, 13^{th} week

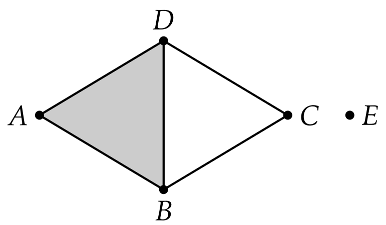
1. For a given simplicial complex K the function G is defined by the values in the following array.

σ	A	B	C	D	E	AB	AD	BC	BD	CD	ABD
$G(\sigma)$	3	2	0	3	2	6	4	1	4	1	5

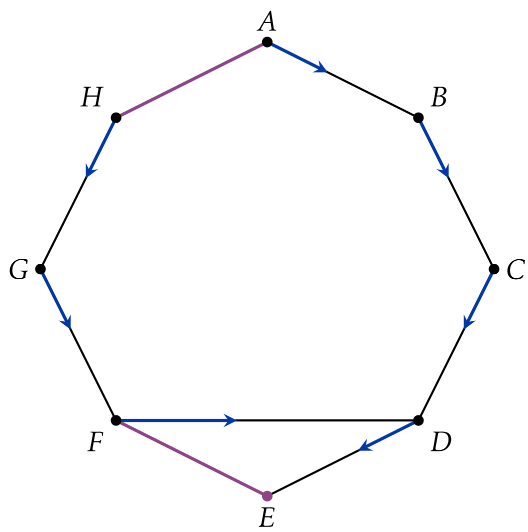
   (a) Show that G is discrete Morse function on K.

   (b) Determine the critical simplices and draw the corresponding vector field V_G.

   (c) Find all non-trivial gradient paths and use cancellation to obtain a new vector field with the minimal possible number of critical simplices.

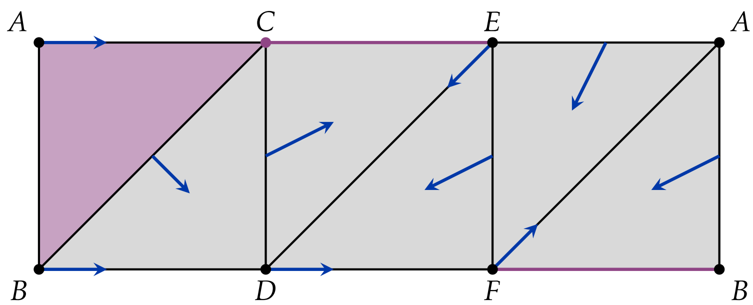


2. Use the given discrete vector field to compute the homology of this simplicial complex.

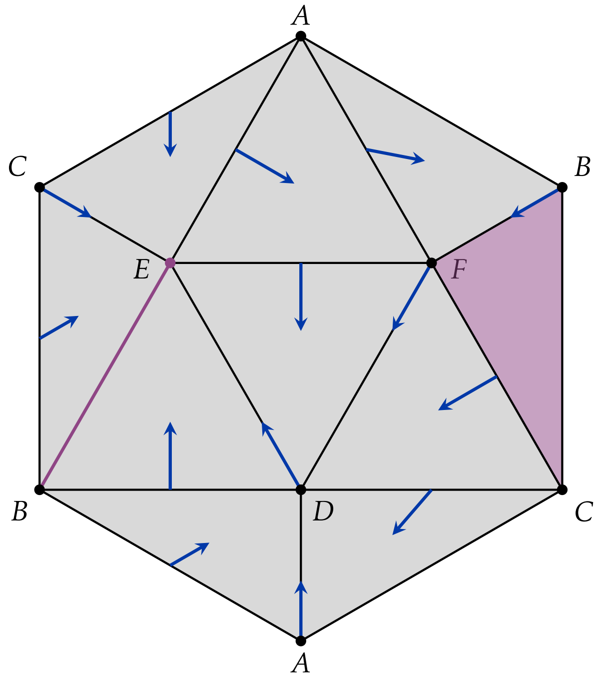


3. Using the triangulations given in the following problems, construct an example of a discrete Morse function with a minimal number of critical simplices for the cylinder and the projective plane.

4. Use the given discrete vector field on the cylinder to compute its homology.



5. Use the given discrete vector field on the projective plane $\mathbb{R}P^2$ to compute its homology.



A function $F\colon K \to \mathbb{R}$ is a **discrete Morse function** on a simplicial complex K if for every $\alpha^{(p)} \in K$

- the set $\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\}$ contains at most one element and
- the set $\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\}$ contains at most one element.

It can be shown that at least one of these sets must be empty. A simplex $\alpha^{(p)}$ is called **critical** if both sets are empty.

A **discrete vector field** V on a simplicial complex K is a collection of pairs

$$\{\alpha^{(p)} < \beta^{(p+1)}\}$$

of simplices in K such that each simplex is in at most one pair of V. We can represent V graphically by drawing an arrow from the centre of α to the centre of β for each pair $\{\alpha < \beta\}$ in V. Every Morse function F defines a discrete vector field V_F with $(\alpha^{(p)}, \beta^{(p+1)}) \in V_F$ if and only if $F(\alpha) \geq F(\beta)$.

A V**-path** is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}$$

such that for each $i \in \{0, 1, \ldots, r\}$ the pair (α_i, β_i) is in V and $\beta_i > \alpha_{i+1}$ with $\alpha_{i+1} \neq \alpha_i$.
The path is **closed** if $\alpha_0 = \alpha_{r+1}$.
A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed V-paths.
If V is the gradient vector field of a discrete Morse function F, then

$$F(\alpha_0) \geq F(\beta_0) \geq F(\alpha_1) \geq F(\beta_1) \geq \ldots \geq F(\beta_r) \geq F(\alpha_{r+1}).$$

If F is a discrete Morse function on K with critical simplices $\beta^{(p+1)}$ and $\alpha^{(p)}$ and there is exactly one gradient path from $\partial\beta$ to α, then there is another Morse function G on K with the same critical simplices except α and β. The gradient vector field of G is equal to that of F except along the unique gradient path from β to α, where the field is reversed.

To construct a discrete Morse function F on a 1-dimensional simplicial complex K:

- find a spanning tree of K,
- pick a vertex v_0 and set $F(v_0) = 0$,
- for all other vertices v in K let $F(v)$ be twice the distance to the root v_0 along the unique path in the spanning tree,
- for every edge uv contained in the spanning tree let $F(uv) = \frac{1}{2}(F(u) + F(v))$,
- for all edges uv not in the spanning tree choose $F(uv)$ greater than the maximal value of F on the spanning tree.

All edges not in the spanning tree and v_0 are critical simplices of F.

To construct a discrete Morse function F on a simplicial complex K from a given discrete vector field V:

- let $F(\sigma) = \dim(\sigma)$ for all critical simplices σ of V,
- for every directed path starting in a critical p-simplex assign descending values from the interval $(p, p-1)$ until reaching a simplex τ with $F(\tau)$ already assigned, then if $F(\tau)$ is bigger than the value of F on the previous simplex in the path, assign a new (lower) value from $(p, p-1)$ to τ and appropriate new values along all directed paths continuing from τ.

If L is a sub-complex of K, then any discrete Morse function F on L can be extended to a discrete Morse function G on K by

$$G(\sigma) = \begin{cases} F(\sigma), & \sigma \in L, \\ \dim(\sigma) + \max_{\tau \in L} F(\tau), & \sigma \in K \setminus L \end{cases}$$

This is not very efficient since every simplex in $K \setminus L$ is critical. If K is collapsible to L, we can improve on this. If $K = L \cup \{\alpha^{(p)}, \beta^{(p+1)}\}$ and α is a free face of β, then we can define

$$G(\sigma) = \begin{cases} F(\sigma), & \sigma \in L, \\ \max_{\tau \in L} F(\tau) + 1, & \sigma = \beta, \\ \max_{\tau \in L} F(\tau) + 2, & \sigma = \alpha. \end{cases}$$

It is easy to see that G is a Morse function on K that extends F and the two simplices α and β are not critical. Using this we can extend any Morse function one collapsible pair at a time, only adding critical simplices when we run out of collapsible pairs (ie. when homology of the complex changes).

Let K be a simplicial complex of dimension n and choose a discrete gradient vector field on K. For each p let M_p be the vector space with $\mathbb{Z}_2$ coefficients, spanned by critical p-simplices. The **Morse chain complex** is the sequence

$$0 \to M_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{p+1}} M_p \xrightarrow{\partial_p} M_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} M_0 \to 0.$$

The boundary homomorphism $\partial_p \colon M_p \to M_{p-1}$ on critical p-simplex σ is given by

$$\partial_p(\sigma^p) = \sum_i \alpha_i \tau_i^{p-1},$$

where α_i is the number of gradient paths (mod 2) starting from the boundary of σ and ending in τ_i. This is extended to chains as

$$\partial_p\left(\sum_i a_i \sigma_i\right) = \sum a_i \partial_p(\sigma_i).$$

If $Z_p(M) = \ker \partial_p$ and $B_p(M) = \mathrm{im} \partial_{p+1}$, then the **Morse homology** of the simplicial complex K with $\mathbb{Z}_2$ coefficients is

$$H_p(M;\mathbb{Z}_2) = \frac{Z_p(M)}{B_p(M)}.$$

The Morse homology groups are isomorphic to the simplicial homology groups, $H_p(M;\mathbb{Z}_2) \cong H_p(K;\mathbb{Z}_2)$, and the Betti numbers of K are $b_p(K;\mathbb{Z}_2) = \mathrm{rank} H_p(M;\mathbb{Z}_2)$.

Let c_p denote the number of critical simplices of K in dimension p and b_p the Betti numbers of K and let $n = \dim K$.

The Weak Morse inequalities.

For $p = 0, 1, 2, \ldots, n$ we have $c_p \geq b_p$ and

$$\chi(K) = c_0 - c_1 + c_2 - \ldots + (-1)^n c_n = b_0 - b_1 + b_2 - \ldots + (-1)^n b_n.$$

The Strong Morse inequalities.

For $p = 0, 1, 2, \ldots, n, n+1$ we have

$$c_p - c_{p-1} + c_{p-2} - \ldots + (-1)^p c_0 \geq b_p - b_{p-1} + \ldots + (-1)^p b_0.$$

To compute Morse homology with $\mathbb{Z}$ coefficients let M_p be the vector space with $\mathbb{Z}$ coefficients, spanned by critical p-simplices with the corresponding Morse chain complex

$$0 \to M_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{p+1}} M_p \xrightarrow{\partial_p} M_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} M_0 \to 0.$$

We define an inner product $\langle \cdot, \cdot \rangle$ on M_p on generators of M_p as

$$\langle \sigma_1, \sigma_2 \rangle = \begin{cases} 0, & \sigma_1 \neq \sigma_2, \\ 1, & \sigma_1 = \sigma_2, \end{cases}$$

and extend it linearly to all elements of M_p.

For a gradient path

$$\gamma \colon \sigma_0,\, \tau_0,\, \sigma_1,\, \tau_1,\, \ldots,\, \tau_{r-1},\, \sigma_r,$$

we define the **multiplicity** of γ, $m(\gamma)$, as

$$m(\gamma) = \prod_{i=0}^{r-1} \langle \partial\tau_i, \sigma_i \rangle \langle \partial\tau_i, \sigma_{i+1} \rangle.$$

The multiplicity of γ is 1 if the orientation of σ_r agrees with the orientation obtained by sliding σ_0 along γ and -1 if the orientation is reversed. For any 1-path γ we have $m(\gamma) = 1$. The multiplicity of the constant path is also 1.
Let $\Gamma(\sigma, \sigma')$ denote the set of all gradient paths from σ to σ'. For critical simplices $\tau^{(p+1)}$ and $\sigma^{(p)}$ we define

$$\langle \partial\tau, \sigma \rangle = \sum_{\sigma' < \tau} \left(\langle \partial\tau, \sigma' \rangle \sum_{\gamma \in \Gamma(\sigma', \sigma)} m(\gamma) \right).$$

If $M_{p+1} = \langle \tau_1, \ldots, \tau_n \rangle$ and $M_p = \langle \sigma_1, \ldots, \sigma_m \rangle$, then the entries of the boundary matrix $D_p \colon M_{p+1} \to M_p$ are

$$(D_p)_{i,j} = \langle \partial\tau_i, \sigma_j \rangle.$$

We obtain Morse homology groups with $\mathbb{Z}$ coefficients as

$$H_p(M;\mathbb{Z}) = \frac{\ker D_p}{\operatorname{im} D_{p+1}}.$$