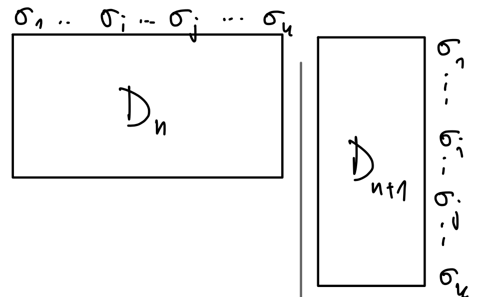
simultaneous column/row reduction on D_n / D_{n+1}

$$
\begin{array}{ccccc}
 & D_{n+1} & & D_n & \\
 & \Updownarrow & & \Updownarrow & \\
C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \\
 & & \shortparallel & & \\
 & & \langle \sigma_1, \ldots, \sigma_i, \ldots, \sigma_j, \ldots, \sigma_k \rangle & &
\end{array}
$$

$\partial_n \partial_{n+1} = 0$ or $D_n D_{n+1} = 0$



$\sigma_i \leftrightarrow \sigma_j$	$c_i \leftrightarrow c_j$	$r_i \leftrightarrow r_j$
$\sigma_i \leftarrow \alpha \sigma_i$	$c_i \leftarrow \alpha c_i$	$r_i \leftarrow \frac{1}{\alpha} r_i$
$\sigma_i \leftarrow \sigma_i + \alpha \sigma_j$	$c_i \leftarrow c_i + \alpha c_j$	$r_j \leftarrow r_j - \alpha r_i$

$C_3 = 0$

$C_2 = \langle ABC, ABD, ACD, BCD \rangle$

$C_1 = \langle AB, AC, AD, BC, BD, BE, CD, CE \rangle$

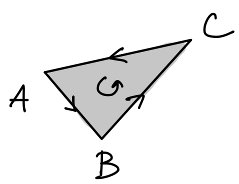
$C_0 = \langle A, B, C, D, E \rangle$

$\partial_3 : \underset{C_3}{0} \longrightarrow C_2, \quad \partial_3 = 0$

$\partial_2 : C_2 \longrightarrow C_1$

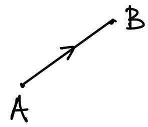
$\partial_1 : C_1 \longrightarrow C_0$

$\partial_0 : C_0 \longrightarrow 0, \quad \partial_0 = 0$



$\partial_2 : C_2 \longrightarrow C_1$

$\partial_2(ABC) = AB + BC - AC$



$\partial_2 : C_1 \longrightarrow C_0$

$\partial_2(AB) = B - A$

$$
D_2 = \begin{array}{c} \\ AB \\ AC \\ AD \\ BC \\ BD \\ BE \\ CD \\ CE \end{array}
\begin{array}{c}
\begin{array}{cccc} ABC & ABD & ACD & BCD \end{array} \\
\begin{bmatrix}
1 & 1 & & \\
-1 & & 1 & \\
 & -1 & -1 & \\
1 & & & 1 \\
 & 1 & & -1 \\
 & & & \\
 & & 1 & 1 \\
 & & &
\end{bmatrix}
\end{array}
\qquad
D_1 = \begin{array}{c|cccccccc}
 & AB & AC & AD & BC & BD & BE & CD & CE \\ \hline
A & -1 & -1 & -1 & & & & & \\
B & 1 & & & -1 & -1 & -1 & & \\
C & & 1 & & 1 & & & -1 & -1 \\
D & & & 1 & & 1 & & 1 & \\
E & & & & & & 1 & & 1
\end{array}
$$

Also: $D_0 = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$ and $D_3 = \begin{array}{c} ABC \\ ABD \\ ACD \\ BCD \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Our "simultaneous reduction" will start from bottom up (i.e., with the pair D_0, D_1). Each (D_n, D_{n+1})-stage will consist of two main steps:

- Column-reduce D_n into a 'lower-triangular normal form' and do corresponding operations on the rows of D_{n+1}. We end up with matrices D_n' and D_{n+1}'.

- Row-reduce D_{n+1}' into a 'upper-triangular normal form'. We end up with D_{n+1}''.

(We will keep track of the generators at each step...)

Next stage starts with (D_{n+1}'', D_{n+2}).

Let's start with (D_0, D_1):

$$D_0 = \begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \longleftarrow \text{ no column reduction needed, go directly to } D_1:$$

$$D_1 = \begin{array}{c|cccccccc} & AB & AC & AD & BC & BD & BE & CD & CE \\ \hline A & -1 & -1 & -1 & & & & & \\ B & 1 & & & -1 & -1 & -1 & & \\ C & & 1 & & 1 & & & -1 & -1 \\ D & & & 1 & & 1 & & 1 & \\ E & & & & & & 1 & & 1 \end{array} \longrightarrow \underbrace{\begin{array}{c|cccccccc} & AB & AC & AD & BC & BD & BE & CD & CE \\ \hline B-A & +1 & +1 & +1 & & & & & \\ C-B & 0 & +1 & +1 & +1 & +1 & +1 & & \\ E-C & & 0 & +1 & 0 & +1 & +1 & +1 & +1 \\ E-D & & & 0 & 0 & 0 & +1 & 0 & +1 \\ E & & & & & & 0 & 0 & 0 \end{array}}_{D_1''}$$

new basis "vectors" of $C_0(X) = \ker \partial_0$

This already gives a nice description of $H_0(X; \mathbb{Q})$:

$$H_0(X;\mathbb{Q}) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle B-A, C-B, E-C, E-D, E\rangle}{\underbrace{\langle B-A, C-B, E-C, E-D\rangle}_{\text{basis vectors of } \operatorname{im}\partial_1 \text{ (the ones corresponding to pivot rows of } D_1')}} = \langle [E] \rangle \cong \mathbb{Q}.$$

Let's continue with (D_1'', D_2):

column reduction on D_1''

$D_1'' =$

$$\begin{array}{c|cccccccc|}
 & AB & AC & AD & BC & BD & BE & CD & CE \\
\hline
B-A & 1 & 1 & 1 & & & & & \\
C-B & 0 & 1 & 1 & 1 & 1 & 1 & & \\
E-C & & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
E-D & & & 0 & 0 & 0 & 1 & 0 & 1 \\
E & & & & & & 0 & 0 & 0
\end{array}$$

$\downarrow$

$$\begin{array}{c|cccccccc|}
 & AB & AC-AB & AD-AB & BC & BD & BE & CD & CE \\
\hline
B-A & 1 & 0 & 0 & & & & & \\
C-B & 0 & 1 & 1 & 1 & 1 & 1 & & \\
E-C & & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
E-D & & & & 0 & 0 & 1 & 0 & 1 \\
E & & & & & & 0 & 0 & 0
\end{array}$$

$\downarrow$

$$\begin{array}{c|cccccccc|}
 & AB & AC-AB & AD-AC & BC-AC+AB & BD-AC+AB & BE-AC+AB & CD & CE \\
\hline
B-A & 1 & 0 & 0 & & & & & \\
C-B & 0 & 1 & 0 & 0 & 0 & 0 & & \\
E-C & & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
E-D & & & & 0 & 0 & 1 & 0 & 1 \\
E & & & & & & 0 & 0 & 0
\end{array}$$

$\downarrow$

$$\begin{array}{c|cccccccc|}
 & AB & AC-AB & AD-AC & BC-AC+AB & BD-AD+AB & BE-AD-AB & CD-AD+AC & CE-AD+AC \\
\hline
B-A & 1 & 0 & 0 & & & & & \\
C-B & 0 & 1 & 0 & 0 & 0 & 0 & & \\
E-C & & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
E-D & & & & 0 & 0 & 1 & 0 & 1 \\
E & & & & & & 0 & 0 & 0
\end{array}$$

$\downarrow$

$$\begin{array}{c|cccccccc|}
 & AB & AC-AB & AD-AC & BC-AC+AB & BD-AD+AB & BE-AD-AB & CD-AD+AC & CE-AD+AC \\
\hline
B-A & 1 & 0 & 0 & & & & & \\
C-B & 0 & 1 & 0 & 0 & 0 & 0 & & \\
E-C & & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
E-D & & & & 0 & 0 & 1 & 0 & 1 \\
E & & & & & & 0 & 0 & 0
\end{array}$$

$\longrightarrow$

$$\begin{array}{c|cccccccc|}
 & AB & AC-AB & AD-AC & BC-AC+AB & BD-AD+AB & BE-AD-AB & CD-AD+AC & CE-BE+AC-AB \\
\hline
B-A & 1 & 0 & 0 & & & & & \\
C-B & 0 & 1 & 0 & 0 & 0 & 0 & & \\
E-C & & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
E-D & & & & 0 & 0 & 1 & 0 & 0 \\
E & & & & & & 0 & 0 & 0
\end{array}$$

Also perform the operations on D_2 simultaneously with operations on D_1''. (We do them "in-place" and skip the notation of the generators..)

$D_2'' =$

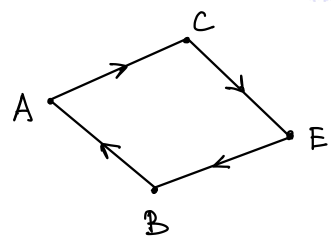
$$\begin{array}{c|cccc|}
 & ABC & ABD & ACD & BCD \\
\hline
AB & \cancel{1}0 & \cancel{1}0 & \cancel{1}0 & \\
\cancel{AC} & -\cancel{1}0 & -\cancel{1}0 & \cancel{1}0 & \cancel{1}0 \\
\cancel{AD} & & -\cancel{1}0 & -\cancel{1}0 & -\cancel{1}0 \\
\cancel{BC} & 1 & & & 1 \\
\cancel{BD} & & 1 & & -1 \\
\cancel{BE} & & & & \\
\cancel{CD} & & & 1 & 1 \\
\cancel{CE} & & & &
\end{array}$$

Luckily, we can skip row reduction on this matrix (only row reordering is required).

Hence:

$$H_1(X;\mathbb{Q}) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle BC-AC+AB,\ BD-AD+AB,\ CD-AD+AC,\ CE-BE+AC-AB\rangle}{\langle BC-AC+AB,\ BD-AD+AB,\ CD-AD+AC\rangle} =$$

$$= \langle [CE-BE+AC-AB] \rangle \cong \mathbb{Q}.$$



Finish with column reduction on D_2'':

$$
\begin{array}{cccc}
ABC & ABD & ACD & BCD \\
\end{array}
$$
$$
\begin{bmatrix}
 & & & \\
 & & & \\
 & & & \\
1 & & & 1 \\
 & 1 & & -1 \\
 & & & \\
 & & 1 & 1
\end{bmatrix}
$$

$$\downarrow$$

$$
\begin{array}{cccc}
ABC & ABD & ACD & \\
\end{array}
$$
$$
\begin{bmatrix}
 & & & \\
 & & & \\
 & & & \\
1 & & & 0 \\
 & 1 & & 0 \\
 & & & \\
 & & 1 & 0
\end{bmatrix}
$$

a generator for $\ker \partial_2$

$BCD - ABC + ABD - ACD$

Since $\partial_3 = 0$ we have $\operatorname{im} \partial_3 = 0$, so

$$H_2(X;\mathbb{Q}) = \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \frac{\langle BCD - ABC + ABD - ACD \rangle}{0} = \langle [BCD - ABC + ABD - ACD] \rangle \cong \mathbb{Q}.$$

(Geometrically, the generator is the boundary of the tetrahedron ABCD.)