Ime in priimek | Vpisna številka

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Osnove matematične analize: drugi kolokvij

7. januar 2026

Čas pisanja je 90 minut. Dovoljena je uporaba 2 listov A4 formata s formulami. Uporaba kalkulatorja ali drugih pripomočkov ni dovoljena. Vse odgovore dobro utemelji!

1. naloga (25 točk)

Funkcija f je podana s predpisom $f(x) = x\ln^2(x)$.

a) (10 točk) Določi definicijsko območje, poišči ničle in limite na robu definicijskega območja.

- $D_{\ln} = (0,\infty) \longrightarrow D_f = (0,\infty) = \mathbb{R}^+$
- $f(x) = x\ln^2 x = 0$

  $x = 0 \notin D_f$

  $\ln x = 0$

  $x = 1$ edina ničla
- $\lim\limits_{x\searrow 0} f(x) = \lim\limits_{x\searrow 0} x\ln^2 x = \lim\limits_{x\searrow 0} \frac{\ln^2 x}{\frac{1}{x}} \overset{LH}{=} \lim\limits_{x\searrow 0} \frac{2\ln x\cdot\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\searrow 0} \frac{2\ln x}{-\frac{1}{x}} \overset{LH}{=} \lim\limits_{x\searrow 0} \frac{2\frac{1}{x}}{\frac{1}{x^2}} = \lim\limits_{x\searrow 0} \frac{2x^2}{x} = \lim\limits_{x\searrow 0} 2x = 0$
- $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x\ln^2 x = \infty$

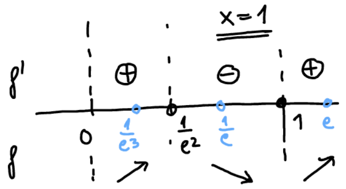
b) (8 točk) Poišči stacionarne točke in območja naraščanja/padanja,

$f'(x) = \ln^2 x + x\cdot 2\ln x\cdot\frac{1}{x} =$

$= \ln^2 x + 2\ln x = \ln x(\ln x + 2) = 0$

$\ln x = 0$, $x = 1$

$\ln x = -2$, $x = \frac{1}{e^2}$



$f'(\frac{1}{e^3}) = -3(-3+2) = 3 > 0$

$f'(\frac{1}{e}) = -1(-1+2) = -1 < 0$

$f'(e) = 1(1+2) = 3 > 0$

$f(\frac{1}{e^2}) = \frac{1}{e^2}\cdot(-2)^2 = \frac{4}{e^2} \doteq 0.54$

$f(1) = 0$

f narašča na $(0,\frac{1}{e^2}) \cup (1,\infty)$

f pada na $(\frac{1}{e^2}, 1)$

$\frac{1}{e^2}$ je (lok.) maksimum, 1 je (lok.) minimum

c) (7 točk) Na katerih intervalih je funkcija f konveksna in kje konkavna?

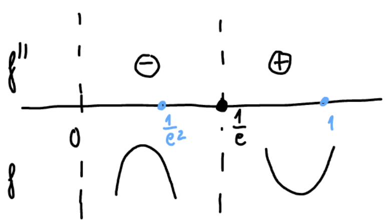
$f''(x) = 2\ln x\cdot\frac{1}{x} + 2\cdot\frac{1}{x} = \frac{2}{x}(\ln x + 1) = 0$

$\ln x = -1$

$x = \frac{1}{e}$

$f''(\frac{1}{e^2}) = 2e^2(-2+1) = -2e^2 < 0$

$f''(1) = 2\cdot 1 = 2 > 0$



f je konveksna na $(\frac{1}{e},\infty)$

f je konkavna na $(0,\frac{1}{e}$

2. naloga (25 točk)

Naj bo

$$f(x,y)=\frac{x}{1+x^2+y^2}.$$

a) (6 točk) Izračunaj smerni odvod funkcije f v točki $(1,2)$ v smeri vektorja $\vec{v}=(1,1)$.

$$f_x=\frac{1(1+x^2+y^2)-x\cdot 2x}{(1+x^2+y^2)^2}=\frac{1-x^2+y^2}{(1+x^2+y^2)^2}\longrightarrow f_x(1,2)=\frac{1-1+4}{(1+1+4)^2}=\frac{4}{36}=\frac{1}{9}$$

$$f_y=\frac{0-x\cdot 2y}{(1+x^2+y^2)^2}=\frac{-2xy}{(1+x^2+y^2)^2}\longrightarrow f_y(1,2)=\frac{-2\cdot 1\cdot 2}{36}=-\frac{4}{36}=-\frac{1}{9}$$

$$(\operatorname{grad} f)(1,2)=\left(\frac{1}{9},-\frac{1}{9}\right)$$

$$f_{(1,1)}(1,2)=\frac{\left(\frac{1}{9},-\frac{1}{9}\right)\cdot(1,1)}{\|(1,1)\|}=\frac{\frac{1}{9}-\frac{1}{9}}{\sqrt{2}}=\underline{\underline{0}}$$

b) (9 točk) Določi stacionarne točke funkcije f

$$\left.\begin{aligned}f_x&=0\\ f_y&=0\end{aligned}\right\}\quad \frac{1-x^2+y^2}{(1+x^2+y^2)^2}=0,\qquad \frac{-2xy}{(1+x^2+y^2)^2}=0$$

$$1-x^2+y^2=0$$
$$-2xy=0$$

i) $x=0$

$1+y^2=0$

$y^2=-1$ //

ni reš.

ii) $y=0$

$1=x^2$

$\underline{\underline{x=\pm 1}}$

$\underline{\underline{T_1(1,0)}}$ $\quad$ $\underline{\underline{T_2(-1,0)}}$

c) (10 točk) Klasificiraj stacionarne točke funkcije f.

$$f_{xx}=\frac{2x(x^2-3(1+y^2))}{(1+x^2+y^2)^3}\qquad f_{xy}=\frac{-2y(1-3x^2+y^2)}{(1+x^2+y^2)^3}\qquad f_{yy}=\frac{-2x(1+x^2-3y^2)}{(1+x^2+y^2)^3}$$

• $(1,0)$

$$\left.\begin{aligned}f_{xx}(1,0)&=\frac{2(1-3)}{8}=-\frac{4}{8}=-\frac{1}{2}\\ f_{xy}(1,0)&=0\\ f_{yy}(1,0)&=\frac{-2\cdot 2}{8}=-\frac{1}{2}\end{aligned}\right\}\quad H_f(1,0)=\begin{bmatrix}-\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{bmatrix},\ \det H_f=\frac{1}{4}>0 \quad \text{maksimum}$$

$\ominus$

• $(-1,0)$

$$\left.\begin{aligned}f_{xx}(-1,0)&=\frac{-2(+1-3)}{8}=\frac{4}{8}=\frac{1}{2}\\ f_{xy}&=0\\ f_{yy}(-1,0)&=\frac{2\cdot 2}{8}=\frac{1}{2}\end{aligned}\right\}\quad H_f(-1,0)=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix},\ \det H_f=\frac{1}{4}>0 \quad \text{minimum}$$

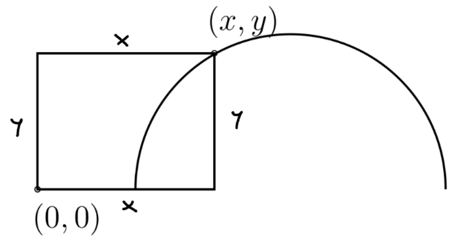
$\oplus$

3. naloga (25 točk)

Med pravokotniki, ki imajo stranice vzporedne koordinatnim osem, eno oglišče v točki $(0,0)$, nasprotno oglišče pa v prvem kvadrantu na krožnici z enačbo

$$(x-2)^2 + y^2 = 2,$$

iščemo tistega, ki ima največji možen obseg.



a) (5 točk) Zapiši obseg pravokotnika kot funkcijo koordinat oglišča (x,y), ki je nasprotno $(0,0)$ (kot na sliki).

$$\sigma(x,y) = 2x+2y$$

b) (5 točk) Zapiši ustrezno Lagrangeovo funkcijo za problem vezanih ekstremov.

$$O(x,y,\lambda) = 2x+2y+\lambda\left((x-2)^2+y^2-2\right)$$

c) (15 točk) Kolikšen je največji možen obseg takega pravokotnika?

$$O_x = 2 + 2\lambda(x-2) = 0 \to 2\lambda(x-2) = -2 \to \lambda(x-2) = -1 \to x-2 = -\frac{1}{\lambda} \to x = -\frac{1}{\lambda}+2$$

$$O_y = 2 + 2\lambda y = 0 \to 2\lambda y = -2 \to \lambda y = -1 \to y = -\frac{1}{\lambda}$$

$$O_\lambda = (x-2)^2+y^2-2 = 0$$

$$\left(-\frac{1}{\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 = 2$$

$$\frac{2}{\lambda^2} = 2$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$\lambda = 1$, $y = -1$, $x = 1$

$(1,-1)$ ni v I. kvadrantu

$\lambda = -1$, $y = 1$, $x = 3$

$(3,1)$ rešitev

$$\sigma(3,1) = 2\cdot 3 + 2\cdot 1 = 8$$

največji možni obseg

4. naloga (25 točk)

a) (17 točk) Izračunaj

$$\int x \arctan(x)\, dx.$$

$$\int x \operatorname{arctg}(x)\, dx = \frac{x^2}{2}\operatorname{arctg}x - \int \frac{1}{2}\,\frac{x^2}{1+x^2}\,dx \overset{*}{=} \frac{x^2}{2}\operatorname{arctg}x - \frac{1}{2}\left(x - \operatorname{arctg}x\right) + C =$$

$u = \operatorname{arctg}x \qquad du = \frac{1}{1+x^2}\,dx$

$dv = x\,dx \qquad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\operatorname{arctg}x - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg}x + C$$

$$= \underline{\underline{\frac{1}{2}(x^2+1)\operatorname{arctg}x - \frac{1}{2}x + C}}$$

$$* \int \frac{x^2}{1+x^2}\,dx = \int \frac{1+x^2-1}{1+x^2}\,dx = \int 1 - \frac{1}{1+x^2}\,dx = x - \operatorname{arctg}x + C$$

b) (8 točk) Izračunaj

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x \cos x}{\sin^2 x + 1}\,dx.$$

$$I = \int_0^1 \frac{t^2\,dt}{t^2+1} = \int_0^1 \frac{t^2+1-1}{t^2+1}\,dt = \int_0^1 1 - \frac{1}{1+t^2}\,dt = \left(t - \operatorname{arctg}t\right)\Big|_0^1 =$$

$t = \sin x$

$dt = \cos x\,dx$

$x = 0 \rightarrow t = 0$

$x = \pi/2 \rightarrow t = 1$

$$= 1 - \overbrace{\operatorname{arctg}1}^{\pi/4} - \left(0 - \overbrace{\operatorname{arctg}0}^{0}\right) = \underline{\underline{1 - \frac{\pi}{4}}}$$