Q14 — Computational Advertising: k-Greedy-BALANCE (15 points)

Setup

- **Query types:** x (served by A or C), y (A only), z (C only)
- **Ads:** A (serves x, y) and C (serves x, z), each with budget \$B, bid \$1
- **Total queries:** 2B. Optimal revenue = \$2B.
- **k-Greedy-BALANCE algorithm:**
  1. If only one ad can serve the query, serve it.
  2. If both can serve (query type x, both have budget):
     - Serve C if unspent(C) ⩾ unspent(A) + k
     - Otherwise serve A

Key constraint: Only configurations where the optimal achieves \$2B revenue.

Part 1 (2 points) — B-Greedy-BALANCE: CR = 1/2

When k = B, the condition unspent(C) ⩾ unspent(A) + B is essentially never satisfied — it requires C to have full budget while A has zero. So the algorithm **always prefers A** for x-queries until A is exhausted.

This reduces to the **Greedy algorithm** (with preference for A). From lecture:

$$\boxed{\text{Competitive ratio of Greedy} = \frac{1}{2}}$$

Part 2 (2 points) — 1-Greedy-BALANCE: CR = 3/4

When k = 1, the condition becomes: serve C whenever unspent(C) > unspent(A). This is exactly the **BALANCE algorithm** — always serve the ad with more remaining budget (ties broken in favor of A).

From lecture, BALANCE with 2 advertisers:

$$\boxed{\text{Competitive ratio of BALANCE (2 ads)} = \frac{3}{4}}$$

Part 3 (5 points) — B/4-Greedy-BALANCE: CR > 1/2

Goal: Show the Greedy worst case (CR = 1/2) is impossible.

Greedy worst case: A fully utilized (revenue B), C completely unutilized (revenue 0), total = B, CR = 1/2.

Proof by contradiction: Assume A is fully utilized and C gets 0 queries.

After A serves its first k = B/4 queries:

- unspent(A) = 3B/4
- unspent(C) = B
- Budget gap: B − 3B/4 = B/4 = k

From this point, any **x-query** triggers serving C (since unspent(C) ⩾ unspent(A) + k). So the remaining 3B/4 queries served by A must all be **y-queries**.

After A is exhausted, the remaining B unserved queries are all y-queries (C can't serve y).

Total y-queries: ⩾ 3B/4 + B = **7B/4**

But the optimal can serve at most B y-queries (limited by A's budget). Having 7B/4 > B y-queries makes optimal revenue of 2B **impossible** — contradiction with our input space constraint.

$$\boxed{\mathrm{CR} > \frac{1}{2}} \qquad \blacksquare$$

Part 4 (6 points) — B/4-Greedy-BALANCE: CR = 11/16

Goal: Find the exact competitive ratio.

Let A be fully utilized (B queries) and C be partially utilized with **d** queries.

Counting y-queries

By the same logic as Part 3, after A's first B/4 queries, subsequent x-queries go to C. So A's remaining budget is filled with y-queries:

$$\text{y-queries in A} \geq B - d - \frac{B}{4} = \frac{3B}{4} - d$$

After A is exhausted, C cannot serve y-queries. The remaining unserved queries = B − d, all of type y:

$$\text{Total y-queries} \geq \left(\frac{3B}{4} - d\right) + (B - d) = \frac{7B}{4} - 2d$$

Applying the optimal revenue constraint

For the optimal to achieve 2B, y-queries ⩽ B:

$$\frac{7B}{4} - 2d \leq B \implies 2d \geq \frac{3B}{4} \implies d \geq \frac{3B}{8}$$

Computing the competitive ratio

Algorithm revenue = B + d ⩾ B + 3B/8 = 11B/8.

$$\boxed{\text{CR} = \frac{11B/8}{2B} = \frac{11}{16}}$$

This bound is tight (achievable with a specific query sequence), so the competitive ratio is exactly 11/16.