CS246 Exam 2025 — Question 11: Submodular Functions (10 points)

Part 1 (2 points) — Greedy by Size vs. Optimal Coverage

Question: Pick $k = 3$ sets from {{0,1,4,9}, {0,6,9}, {1,2,4,8}, {3,5,7}, {9}, {0}}. Does greedy-by-size (largest set first) achieve the best coverage?

Answer: No.

Optimal: Pick {0,6,9}, {1,2,4,8}, {3,5,7} → covers **all 10 elements** (the entire universe).

Greedy-by-size: Picks the two size-4 sets first: {0,1,4,9} and {1,2,4,8}. These overlap on {1, 4}, wasting coverage. The third pick can cover at most 9 elements (best case) or 7 (worst case).

The problem is that greedy-by-size selects based on **total size**, not on how many **new** elements are added. The correct greedy algorithm for maximum coverage picks the set with the largest **marginal gain** at each step — that version gives the $(1 - 1/e)$ approximation guarantee for submodular functions.

Part 2 — Submodularity Tests

Recall: F is submodular if for all $A \subseteq B$ and any C: $F(A \cup C) - F(A) \geq F(B \cup C) - F(B)$ (diminishing returns).

Part 2a (2 points) — $F(A) = |A|$

Yes, submodular.

The marginal gain simplifies: $F(A \cup C) - F(A) = |C \setminus A|$ and $F(B \cup C) - F(B) = |C \setminus B|$. Since $A \subseteq B$, we have $C \setminus B \subseteq C \setminus A$, so $|C \setminus A| \geq |C \setminus B|$. ✓

(In fact, this is **modular** — each element independently contributes exactly 1.)

Part 2b (2 points) — $F(A) = \max(A)$, $\max(\emptyset) = 0$

No, not submodular.

Counterexample: $A = \emptyset$, $B = \{1\}$, $C = \{-5\}$.

- LHS: $F(A \cup C) - F(A) = \max(\{-5\}) - 0 = \mathbf{-5}$
- RHS: $F(B \cup C) - F(B) = \max(\{1, -5\}) - 1 = \mathbf{0}$

Is $-5 \geq 0$? **No.** The $\max(\emptyset) = 0$ convention creates trouble when adding negative elements to the empty set.

Part 2c (2 points) — $F(A) = \sum_{i \in A} i^2$

Yes, submodular.

Same argument as Part 2a: the marginal gain is $\Sigma_{i \in C \setminus A}\, i^2$ vs. $\Sigma_{i \in C \setminus B}\, i^2$. Since $A \subseteq B$, we have $C \setminus B \subseteq C \setminus A$, and since $i^2 \geqslant 0$, summing over a superset gives a larger result. ✓

(Also modular — each element independently contributes its non-negative weight i^2.)

Part 2d (2 points) — $F(A) = (\Sigma_{i \in A}\, i)^2$

No, not submodular.

Counterexample: $A = \{1\}$, $B = \{1, 2\}$, $C = \{3\}$.

- LHS: $F(\{1,3\}) - F(\{1\}) = 4^2 - 1^2 = \mathbf{15}$
- RHS: $F(\{1,2,3\}) - F(\{1,2\}) = 6^2 - 3^2 = \mathbf{27}$

Is $15 \geqslant 27$? **No.** This function exhibits **increasing returns** — the marginal gain of adding element c is $2c \cdot \Sigma_{i \in A}\, i + c^2$, which *grows* with the existing sum. Squaring is convex, the opposite of the concavity that submodularity requires.

Summary

Function	Submodular?	Key Reason
$\lvert A \rvert$	**Yes**	Modular; each element contributes 1 independently
$\max(A)$	**No**	$\max(\emptyset) = 0$ convention breaks diminishing returns for negative elements
$\Sigma\, i^2$	**Yes**	Modular with non-negative weights i^2
$(\Sigma\, i)^2$	**No**	Convex/increasing returns — marginal gain grows with set size