Last time

- Matrix & vector norms
- condition number
- Gaussian elimination

Solving $Ax=b$

GE illustration

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 3 & -2 \\ -3 & -4 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ -2 & 3 & -2 & 1 \\ -3 & -4 & 4 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & -2 & 8 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 5 \end{array}\right]$$

Back substitution

Idea :

Multiplication from left $\Rightarrow$ row operations

Multiplication from right $\Rightarrow$ column operations

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 3 & -2 \\ -3 & -4 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 2 \\ -2 & 3 & -2 \\ -3 & -4 & 4 \end{bmatrix} x = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} \leftarrow \text{1st row unchanged}$$

2nd row $\rightarrow$ $2a_1^T + a_2^T$

no change

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}$$

Multiplication from the right $\Rightarrow$ column op.

$$\left[\begin{array}{c:c:c} a_1 & a_2 & a_3 \end{array}\right] \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{Matrices can encode column \& row operations}}$

Algorithm : GE to upper triangular matrix

for $k = 1 \dots n-1$

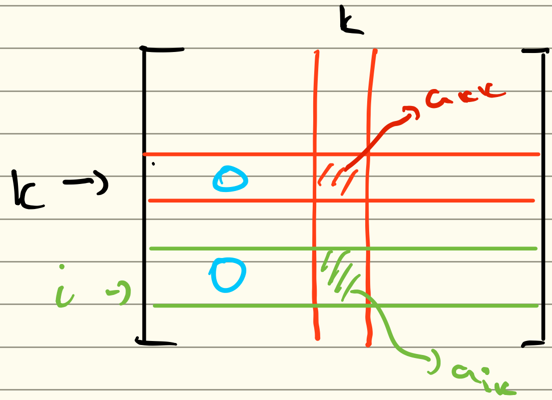
  for $i = k+1 \dots n$

    $\alpha_i = a_{ik} / a_{kk}$

    $a_{ik} = 0$

    for $j = k+1 \dots n$

      $a_{ij} = a_{ij} - \alpha_i a_{kj}$ } vector subtraction



Check $j=k$

$$a_{ik} = a_{ik} - \alpha_i a_{kk}$$

$$= a_{ik} - \frac{a_{ik}}{a_{kk}} \cdot a_{kk}$$

$$= 0$$

of operations

k-th step

- There are $n-k$ rows below k
- 1 multiplication & 1 subtraction (on $n-k$ columns)

Together gives $2(n-k)^2$ operations per k

$$\sum_{k=1}^{n-1} 2(n-k)^2 \underset{j=n-k}{=} 2\sum_{j=1}^{n-1} j^2 = \frac{2(n-1)n(2n-1)}{6}$$

$$\frac{n(2n^2-2n-n+1)}{3}$$

$$\approx \frac{2}{3}n^3 + O(n^2)$$

Need to count divisions (α_i)

$$\sum_{k=1}^{n-1}(n-k) = \sum_{j=1}^{n-1} j = \frac{n(n-1)}{2} \approx O(n^2)$$

$\Rightarrow$ Answer $\boxed{\frac{2}{3}n^3 + O(n^2)}$

Cost of Back Substitution

Solve $Ux = c$

$x_n = c_n / u_{nn}$

for $i = n-1 \dots 1$

$\quad s_i = c_i$

$\quad$ for $j = i+1 \dots n$

$\quad\quad s_i = s_i - u_{ij} \cdot x_j$

$\quad x_i = s / u_{ii}$

\# of operations

$$1 + \sum_{i=1}^{n-1}\left(\left(\sum_{j=i+1}^{n-1} 2\right) + 1\right) = 1 + \sum_{i=1}^{n-1} 2(n-i) + (n-1)$$

$$= 2\sum_{i=1}^{n-1}(n-i) + n$$

$$= 2\sum_{k=1}^{n-1} k + n$$

$$= \cancel{2}\frac{n(n-1)}{\cancel{2}} + n$$

$$= n^2 - n + n$$

$$\boxed{= n^2}$$

Idea from before

Row operation

$$\tilde{A} = A + \alpha_i E_{ij} A \qquad E_{ij} : e_{kl} = \begin{cases} 1 & k=i \ \ l=j \\ 0 & \text{else} \end{cases}$$

$$\hat{A} = (I + \alpha_i E_{ij}) A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \underset{A}{\begin{bmatrix} 1 & -2 & 2 \\ -2 & 3 & -2 \\ -3 & -4 & 4 \end{bmatrix}} x = b$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) A$$

E_{ij} encodes adding i-th row to j-th row

Full step

$$A \mapsto \left(I_n - \frac{a_{21}}{a_{11}} E_{21}\right) A$$

$$\mapsto \left(I_n - \frac{a_{31}}{a_{11}} E_{31}\right)\left(I_n - \frac{a_{21}}{a_{11}} E_{21}\right) A$$

$$\mapsto \left(I_n - \frac{a_{n1}}{a_{11}} E_{n1}\right) \cdots \left(I_n - \frac{a_{21}}{a_{11}} E_{21}\right) A : A^{(1)}$$

Multiply out

$$A^{(1)} = \big(\underbrace{I_n - \frac{a_{n1}}{a_{11}} E_{n1} - \dots - \frac{a_{21}}{a_{11}} E_{21}}_{L_1}\big) A$$

upper triangular

E_{ij} $i>j$

why $E_{i1} \cdot E_{j1} = 0$ since $i, j > 1$

Next iteration

$$L_2 = I_n - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} E_{32} - \dots - \frac{a_{n2}^{(1)}}{a_{22}^{(1)}} E_{n2}$$

$$A^{(2)} = L_2 \cdot A^{(1)} = L_2 L_1 A$$

Claim:

$$\boxed{U = L_{n-1} \cdots L_1 A}$$

We want to now express A

$$L_1^{-1} = I_n + \frac{a_{21}}{a_{11}} E_{21} + \dots + \frac{a_{n1}}{a_{11}} E_{n1}$$

$$L_2^{-1} = I_n + \frac{a_{32}^{(1)}}{a_{22}^{(1)}} E_{32} + \dots -$$

$$\vdots$$

$$L_{n-1}^{-1} = I_n + \frac{a_{n,n-1}^{(n-2)}}{a_{n-1,n-1}^{(n-2)}} E_{n,n-1}$$

Observe: ① L_i^{-1} is easy to compute from L_i
② L_i^{-1} is lower triangular

Computing A

$$A = \underbrace{L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1}}_{L} U$$

$$L = I_n + \frac{a_{21}}{a_{11}} E_{21} + \dots + \frac{a_{n1}}{a_{11}} E_{n1} + \frac{a_{32}^{(1)}}{a_{22}^{(1)}} E_{32} + \dots + \frac{a_{n,n-1}^{(n-2)}}{a_{n-1,n-1}^{(n-2)}} E_{n,n-1}$$

Computing the LU decomposition

for $k = 1, \dots, n-1$

  for $k+1, \dots n$

    $l_{ik} = a_{ik}/a_{kk}$

    for $j = k+1 \dots n$

      $a_{ij} = a_{ij} - l_{ik} a_{kj}$

$u_{ij} = a_{ij}$ (at the end)

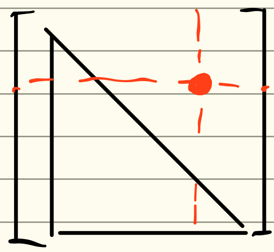
# of operations $\frac{2}{3} n^3 + O(n^2)$ (same as GE)

forward substition $Ly = b$

↳ cost is n^2

Alternate argument for $L_1 \cdot L_2$ is lower diagonal

$$\begin{bmatrix} l_{11} & & 0 \\ \vdots & \ddots & \\ l_{n1} & \cdots & l_{nn} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & & 0 \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} =$$



Solving $Ax=b$ with LU decomposition

① Compute $A = LU$ $\quad \left(\frac{2}{3}n^3 + O(n^2)\right)$

② Solve $Ly = b$ $\quad (n^2 - n)$ ↳ because we have 1 on the diagonal

③ Solve $Ux = y$ $\quad (n^2)$

Example

$$A = \begin{pmatrix} 2 & 1 & 3 & -4 \\ -4 & -1 & -4 & 7 \\ 2 & 3 & 5 & -3 \\ -2 & -2 & -7 & 9 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ -14 \\ 7 \\ -16 \end{pmatrix}.$$

1. $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 1 & 3 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$
2. *Rešimo* $Ly = b$ *in dobimo* $y = (8 \quad 2 \quad -5 \quad -1)^T$.
3. *Rešimo* $Ux = y$ *in dobimo* $x = (1 \quad -1 \quad 1 \quad -1)^T$.

Existence & Stability of LU decomposition

Problematic matrix examples

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 10^{-17} & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

if 10^{-17} is below machine precision

if we have the leading $k \times k$ submatrix of $A \in \mathbb{R}^{n \times n}$ (leading principal submatrices)

Theorem

The following 2 statements are equivalent

① LU decomp. exists & is unique

② k-th leading principal submatrices are invertible for $k = 1 \ldots n$

Partial Pivoting

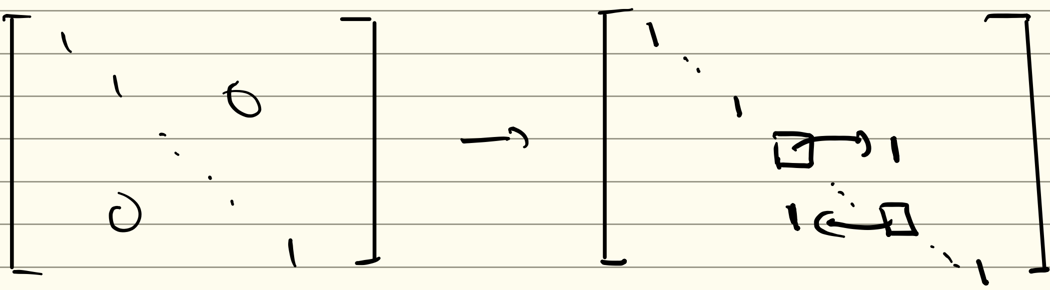
Before elimination of j-th column compare

$$a_{jj}\,;\,a_{j+1,j}\,;\,\dots\,;\,a_{nj}$$

then permute the j-th row with the one who element is the largest (in absolute value)

Exchanging / permuting rows can be done through left multiplication with permutation matrices. (Exchanging j-th row & k-th row)

$$\boxed{P_{jk} = I_n - E_{jj} - E_{kk} + E_{jk} + E_{kj}}$$



Theorem

With partial pivoting $A \mapsto U$ is given by

$$A \mapsto P_1 A \mapsto L_1 P_1 A \mapsto P_2 L_1 P_1 A$$
$$\mapsto L_{n-1} P_{n-1} \cdots L_1 P_1 A := U$$

$$\Rightarrow \boxed{PA = LU}$$

where $P := P_{n-1} P_{n-2} \cdots P_1$ $\quad L = \hat{L}_1^{-1} \hat{L}_2^{-1} \cdots \hat{L}_{n-1}^{-1}$

$\hat{L}_i^{-1} = P^{(i)} L_i^{-1} (P^{(i)})^T$ (See Proof below)

$P^{(i)} = P_{n-1} \cdots P_{i+1}$

Proof

Constructively (above)

$L_{n-1} P_{n-1} \cdots L_1 P_1 A = U$ $\Rightarrow$ move from LHS (left hand side) to RHS (right hand side)

$P_{n-1} \cdots L_1 P_1 A = L_{n-1}^{-1} U$

$L_{n-2} P_{n-2} \cdots L_1 P_1 A = P_{n-1}^T L_{n-1}^{-1} U$

$\vdots$

$A = P_1^T L_1^{-1} P_2^T L_2^{-1} \cdots P_{n-1}^T L_{n-1}^{-1} U$

$PA = \underbrace{P_{n-1} \cdots P_1}_{P} P_1^T L_1^T P_2^T L_2^{-1} \cdots P_{n-1}^T L_{n-1}^{-1} U$

$P \cdot P_1^T = P_{n-1} \cdots P_2 \underbrace{P_1 P_1^T}_{I} = \underbrace{P_{n-1} \cdots P_2}_{P^{(1)}}$

$$PA = P^{(1)} L_1^{T} P_2^{T} \underbrace{(P_3^{T} \cdots P_{n-1}^{T})(P_{n-1} \cdots P_3)}_{I} L_2^{-1} \cdots P_{n-1}^{T} L_{n-1}^{-1} U$$

$$P^{(1)T} \qquad P^{(2)}$$

$$PA = \underbrace{P^{(1)} L_1^{T} P^{(1)T}}_{\hat{L}_1^{-1}} \cdot P^{(2)} L_2^{-1} \cdots L_{n-1}^{-1} U$$

$$\hat{L}_1^{-1} \cdot \quad \cdots \quad \hat{L}_{n-1}^{-1} \cdot U$$

Algorithm

$P, L = I_n$

for $k = 1, \dots, n-1$

find q & k s.t. $|a_{qk}| = \max_{k \le p \le n} |a_{pk}|$

permute rows q & k in A, P, L

for $k+1, \dots n$

$l_{ik} = a_{ik}/a_{kk}$

for $j = k+1 \dots n$

$a_{ij} = a_{ij} - l_{ik} a_{kj}$

$u_{ij} = a_{ij}$ (at the end)

Cost $\frac{2}{3} n^3 + O(n^2)$

pivoting costs $O(n^2)$

rewriting 2 rows $O(n) \to n$ times

Solving $Ax = b$ w/ partial pivoting

1. Compute $PA = LU$ Cost: $\frac{2}{3} n^3 + O(n^2)$
2. Solve $Ly = Pb$ Cost: $n^2 - n$
3. Solve $Ux = y$ Cost: n^2

Theorem: Equivalent statements for matrix A

1. LU decomposition with partial pivoting exists
2. A is invertible

Example

Primer.

$$A = \begin{pmatrix} 2 & 1 & 3 & -4 \\ -4 & -1 & -4 & 7 \\ 2 & 3 & 5 & -3 \\ -2 & -2 & -7 & 9 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ -14 \\ 7 \\ -16 \end{pmatrix}.$$

1. $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{5} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{5} & -\frac{1}{8} & 1 \end{pmatrix}, U = \begin{pmatrix} -4 & -1 & -4 & 7 \\ 0 & \frac{5}{2} & 3 & \frac{1}{2} \\ 0 & 0 & -\frac{16}{5} & \frac{58}{10} \\ 0 & 0 & 0 & \frac{1}{8} \end{pmatrix},$

   $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$
2. Rešimo $Ly = Pb$ in dobimo $y = \begin{pmatrix} -14 & 0 & -9 & -\frac{1}{8} \end{pmatrix}^T$.
3. Rešimo $Ux = y$ in dobimo $x = \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$.

LU razcep z delnim pivotiranjem: koda

Primer: koda

LU w/ complete pivoting

Complete pivoting: find largest entry in $A(j{:}n, j{:}n)$

$\Rightarrow$ exchange rows & columns

Cost is $O(n^3)$ (comparisons & exchanges)

$\rightarrow$ substantially slower (theoretically & practically)

Numerical Stability

Solving $Ax=b$ via LU

3 steps

① $(A+E)=\hat{L}\hat{U}$ (only theoretically unstable step)

② $\hat{L}\hat{y}=b$

③ $\hat{U}\hat{x}=\hat{y}$

Recall: basic rounding error $=2^{-m}=u$

& $|A|=[|a_{ij}|]_{ij}$ (largest abs. value entry)

Theorem: If $A\in\mathbb{R}^{n\times n}$ is invertible w/o pivoting $\hat{L},\hat{U}$ w/ $A=\hat{L}\hat{U}+E$, then

$$|E|\le 3(n-1)u\left(|A|+|\hat{L}||\hat{U}|\right)+O(u^2)$$

Proof: Induction on n

$n=1$ $\quad LU$ is trivial $\Rightarrow |E|=0$ (or just rounding error ($O(u)$))

Assume it holds for $(n-1)\times(n-1)$, write

$$A=\begin{pmatrix}\alpha & w^T\\ v & B\end{pmatrix}$$

holds for this matrix $B=(n-1)\times(n-1)$

$$\hat{A}=\begin{pmatrix}1 & 0\\ \hat{z} & \hat{A}_1\end{pmatrix}$$

$\hat{z}=fl\left(\frac{v}{\alpha}\right)=\frac{v}{\alpha}+f$ ← error

$|f|\le\frac{|v|}{|\alpha|}u$

$$\hat A_1 = fl(B - \hat z w^T) = B - \hat z w^T + F$$

$$\Rightarrow F \le 2u(|B| + |\hat z|\cdot|w^T|) \quad ✱$$

$$\hat A_1 = \hat L_1 \hat U_1 + E_1 \qquad |E_1| \underset{\text{by induction assumption}}{\le} 3(n-2)u\left(|\hat A_1| + |\hat L_1|\cdot|\hat U_1|\right) + O(u^2)$$

$$\hat L \hat U = \begin{pmatrix}1 & 0\\ \hat z & \hat L_1\end{pmatrix}\begin{pmatrix}\alpha & w^T\\ 0 & \hat U_1\end{pmatrix} = \begin{pmatrix}\alpha & w^T\\ \alpha \hat z & \hat z w^T + \hat A_1 - E_1\end{pmatrix}$$

$$= A + \begin{pmatrix}0 & 0\\ \alpha f & \underbrace{F - E_1}_{H}\end{pmatrix}.$$

$$|F - E_1| \le |F| + |E_1|$$

$$✱ \quad \le 2u(|B| + |\hat z|\cdot|w^T|) + 3(n-2)u\left(|\hat A_1| + |\hat L_1|\cdot|\hat U_1|\right) + O(u^2)$$

$$= 3(n-1)u\left(|B| + |\hat z|\cdot|w^T| + |\hat L_1| + |\hat U_1|\right) + O(u^2)$$

from $|A_1| \le (1+2u)(|B| + |\hat z|\cdot|w^T|) + O(u^2)$

So $$|H| \le 3(n-1)u\left(\underbrace{\begin{pmatrix}|\alpha| & |w^T|\\ |v| & |B|\end{pmatrix}}_{|A|} + \underbrace{\begin{pmatrix}1 & 0\\ |\hat z| & |\hat L_1|\end{pmatrix}}_{|\hat L|}\underbrace{\begin{pmatrix}|\alpha| & |w^T|\\ 0 & |\hat U_1|\end{pmatrix}}_{|\hat U|}\right)$$

as required.

With partial pivoting

$$\|E\|_\infty \le 3(n-1)u\left(\|A\|_\infty + (\|\hat{L}\|\,\|\hat{U}\|)_\infty\right) + O(u^2)$$

$$\le 3(n-1)u\left(\|A\|_\infty + (\|\hat{L}\|_\infty \|\hat{U}\|)_\infty\right) + O(u^2)$$

$$\le 3(n-1)u\|A\|_\infty + 3(n-1)(n\|\hat{U}\|_\infty) + O(u^2)$$

→ submultiplicativity

— w/ partial pivoting all elements of L are upper bounded by 1

so $\|L\|_\infty \le n$

Relative error

$$\frac{\|E\|_\infty}{\|A\|_\infty} \le 3(n-1)u + 3(n-1)\cdot nu\,\frac{\|U\|_\infty}{\|A\|_\infty} + O(u^2)$$

↑ stability result partial pivoting

↓ key quantity people use (pivot growth)

Pivot growth

$$\rho(A) = \frac{\max_{i,j}|\hat{u}_{ij}|}{\max_{ij}|a_{ij}|}$$

$$\frac{\|\hat{U}\|_\infty}{\|A\|_\infty} \le n\rho(A)$$

So absolute value of the largest value increases by a factor of 2 (each step)

With partial pivoting pivot growth at most 2^{n-1}

$$\rho(A) \leq 2^{n-1}$$

<u>Proof</u> : $|l_{ij}| \leq 1$ $\quad a_{ij}$ for each step w/ $n-1$ steps

$$a_{ij} \leftarrow a_{ij} - l_{ik} a_{kj}$$

$$a_{ij} = a_{ij} - l_{ik} a_{kj}$$

So absolute value of the largest value increases by a factor of 2 (at most) each step.

But 2^{n-1} is bad (but can happen)

<u>Example</u>

Primer

Matrika

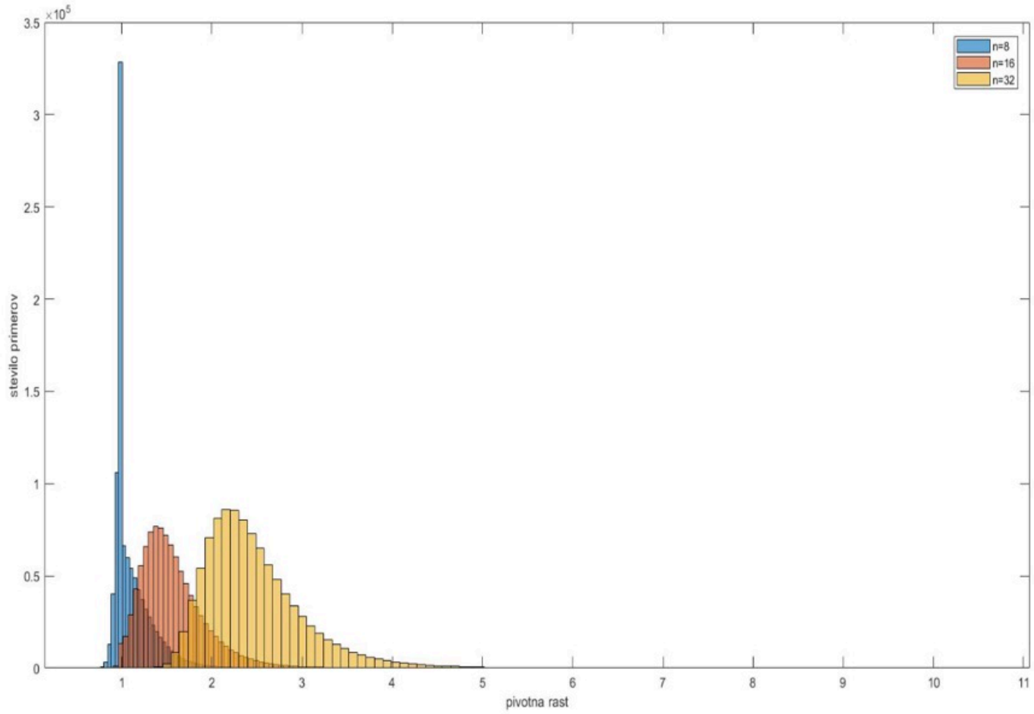
$$A_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ -1 & 1 & \ddots & \vdots & 1 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ -1 & \cdots & \cdots & -1 & 1 \end{pmatrix}$$

ima pivotno rast 2^{n-1}.

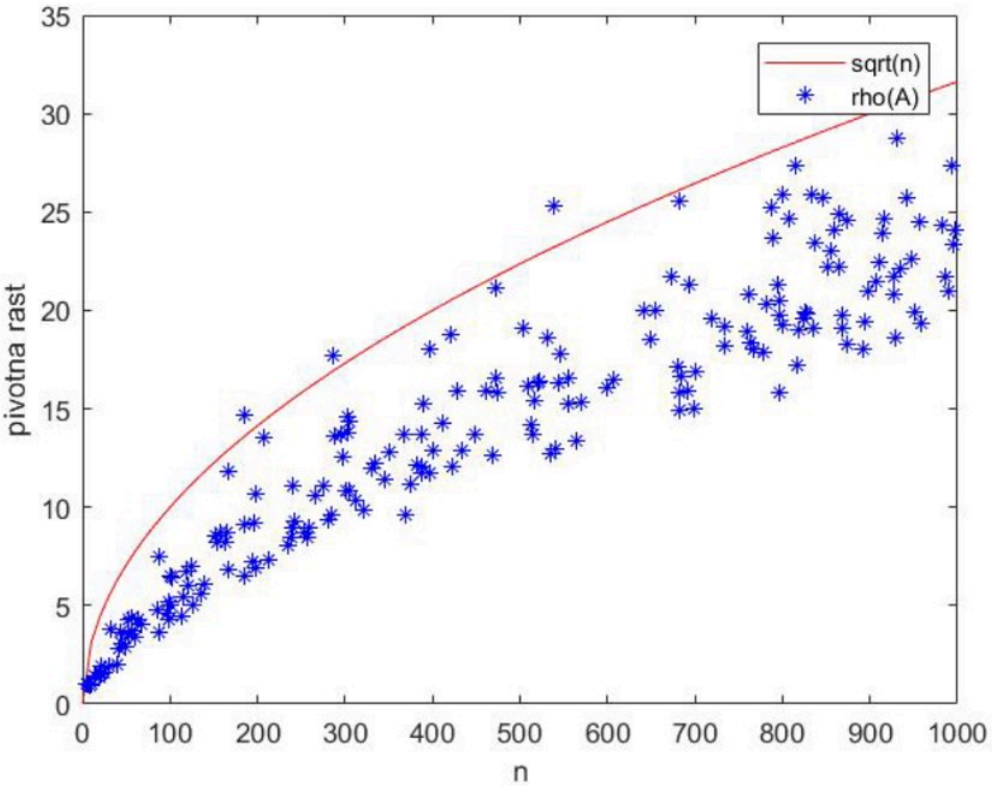
For certain random matrices (uniform entries) $\rho(A) \sim O(n^{1/2})$ ⤳ so more stable

For certain random matrices $\sim O(n)$

Verjetnostne porazdelitve slučajne spremenljivke ρ, generirane z milijon naključnimi matrikami velikosti $n \times n$ (tj. vsak vhod naključen element iz enakomerne zvezne porazdelitve na intervalu $[0, 1]$):



Pivotna rast 200 naključnih matrik velikosti $n \times n$ (tj. vsak vhod naključen element iz enakomerne zvezne porazdelitve na intervalu $[0, 1]$):



LDL decomposition for symmetric matrices

Claim: if $A = A^T$

$$A = LU, \quad U = \underbrace{D}_{\text{diagonal matrix}} L^T \quad \Rightarrow \quad \boxed{A = LDL^T}$$

Proof: $LU = A = A^T = (LU)^T = U^T L^T$

Multiply on the left w/ (L^{-1})

$$L^{-1} L U = L^{-1} (U^T L^T)$$

Multiply on the right w/ $(L^T)^{-1}$

$$U (L^T)^{-1} = L^{-1} U^T L^T \cdot (L^T)^{-1}$$

$$\underbrace{U (L^T)^{-1}}_{\text{2 upper triangular}} = \underbrace{L^{-1} \cdot U^T}_{\text{2 lower triangular}} = D$$

only matrix which is upper & lower triangular
$\Rightarrow$ diagonal

$$U (L^T)^{-1} = D \Rightarrow U = DL^T$$

$$\boxed{A = LU = LDL^T}$$

Cholesky Decomposition

For positive definite matrices

* A is symmetric positive def. iff

$$A^T = A \quad \& \quad x^T A x > 0 \text{ forall } x \neq 0$$

Then there exists a lower triangular matrix V such that

$$\boxed{A = VV^T}$$

where $V = LD^{1/2}$

<u>Proof</u>: $A = LDL^T$

<u>Claim</u>: all diagonal entries of D are strictly positive so $D^{1/2}$ is well-defined

$\to x^T A x > 0$

$\underbrace{x^T L}_{y} D \underbrace{L^T x}_{y} > 0 \Rightarrow y^T D y > 0$

can pick x so $y = e_i$

$\Rightarrow A = LDL^T$

$= LD^{1/2} \cdot D^{1/2} L^T = LD^{1/2} (D^{1/2})^T L^T$

transpose of diagonal matrix is the same

$= LD^{1/2} (LD^{1/2})^T$

$= VV^T$

Algorithm

for $k = 1 \ldots n$

$$v_{kk} = \left(a_{kk} - \sum_{i=1}^{k-1} v_{ki}^2\right)^{1/2}$$

for $j = k+1 \ldots n$

for $i = 1, \ldots, k-1$

$$a_{jk} \leftarrow a_{jk} - v_{ji} v_{ki}$$

$$v_{jk} = a_{jk} / v_{kk}$$

Cost: $\frac{n^3}{3} + O(n^2)$

↑

cheapest computationally

Solving $Ax = b$

① Compute $A = VV^T$ $\left(\frac{n^3}{3} + O(n^2)\right)$

② Solve $Vy = b$ (n^2)

③ Solve $V^T x = y$ (n^2)

$\boxed{\text{Numerically stable}}$