Linearna algebra, vaje, 9. 4. 2026

1. Naj bo A $n \times m$ matrika.

   (a) Označimo z $N(A) \subseteq \mathbb{R}^m$ množico vseh rešitev linearnega sistema $A\mathbf{x} = \mathbf{0}$, tj. $N(A) := \{\mathbf{x} \in \mathbb{R}^m : A\mathbf{x} = \mathbf{0}\}$. Preveri, da je $N(A)$ vektorski podprostor v $\mathbb{R}^m$. Pravimo mu *ničelni prostor matrike A*.

   (b) Označimo s $C(A) \subseteq \mathbb{R}^n$ podmnožico vseh linearnih kombinacij stolpcev matrike A. Preveri, da je $C(A)$ vektorski podprostor v $\mathbb{R}^n$. Pravimo mu *stolpčni prostor matrike A*.

   (c) Konkretno naj bo

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 & 1 \\ 3 & 1 & 3 & 1 & 3 \\ 3 & 3 & 1 & 3 & 3 \\ 3 & 1 & 3 & 1 & 3 \\ 1 & 3 & 3 & 3 & 1 \end{bmatrix}.$$

   Poišči bazi za $N(A)$ in $C(A)$. Ali je vektor $[1,1,1,1,1]^{\mathsf{T}}$ vsebovan v $C(A)$? Če je, ga izrazi v poiskani bazi.

$$C(A) = \{\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \dots + \alpha_m\vec{a}_m : \alpha_i \in \mathbb{R}\}$$
$$= \left\{ [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} : \alpha_i \in \mathbb{R} \right\}$$
$$= \{A\vec{x} : \vec{x} \in \mathbb{R}^m\}$$

c) Naredimo G.e. na A,

prosta spr. (x_4, x_5); (x_1, x_2, x_3)

$$A \to \begin{bmatrix} 1 & 3 & 3 & 3 & 1 \\ 0 & -8 & -6 & -8 & 0 \\ 0 & -6 & -8 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Za $N(A)$ rešujemo $A\vec{x} = \vec{0}$ ...

$x_1 + x_5 = 0$ ... $x_1 = -x_5$

$x_2 + x_4 = 0$ $\quad x_2 = -x_4$

$x_3 = 0$

$$N(A) \ni \vec{x} = \begin{bmatrix} -x_5 \\ -x_4 \\ 0 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

torej je $B_{N(A)} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

prvi trije (prvotni) stolpci A

Iz G.e. razberemo tudi, da je $B_{C(A)} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

Ali je $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in C(A)$? Je, če ima sistem $A\vec{x} = \vec{b}$ rešitev.

$\quad = \vec{b}$

Naredimo G.e. na $[A / \vec{b}]$ .... $\vec{b} = \frac{1}{7}(\vec{a}_1 + \vec{a}_2 + \vec{a}_3)$...

Torej $\vec{b} \in C(A)$.

2. Dana sta vektorja $\mathbf{a} = [1,0,1,-1]^\mathsf{T}$ ter $\mathbf{b} = [0,1,-1,1]^\mathsf{T}$ in podmnožica

$$U := \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{a}\cdot\mathbf{x} = 0 \text{ in } \mathbf{b}\cdot\mathbf{x} = 0\} \subseteq \mathbb{R}^4.$$

(a) Prepričaj se, da je U vektorski podprostor v $\mathbb{R}^4$. Poišči bazo za U in določi njegovo dimenzijo.

(b) Poišči matriki A in B, da bo $U = C(A) = N(B)$.

(c) Ali obstaja 4×4 matrika F, da je $U = C(F) = N(F)$? Če obstaja, jo poišči!

(a) Vzemimo $\vec{x}, \vec{y} \in U$, tj. $\vec{a}\cdot\vec{x} = \vec{a}\cdot\vec{y} = 0$ in $\vec{b}\cdot\vec{x} = \vec{b}\cdot\vec{y} = 0$, ter $\alpha, \beta \in \mathbb{R}$, potem

$$\vec{a}\cdot(\alpha\vec{x} + \beta\vec{y}) = \alpha\underbrace{\vec{a}\cdot\vec{x}}_{0} + \beta\underbrace{\vec{a}\cdot\vec{y}}_{0} = \alpha\cdot 0 + \beta\cdot 0 = 0$$

$$\vec{b}\cdot(\alpha\vec{x} + \beta\vec{y}) = \alpha\underbrace{\vec{b}\cdot\vec{x}}_{0} + \beta\underbrace{\vec{b}\cdot\vec{y}}_{0} = \alpha\cdot 0 + \beta\cdot 0 = 0,$$

torej $\alpha\vec{x} + \beta\vec{y} \in U$ in U je vektorski podprostor v $\mathbb{R}^4$.

Poiščimo bazo za U:

$$\vec{a}^\mathsf{T}\vec{x} = \vec{a}\cdot\vec{x} = 0 \quad \ldots \quad \begin{bmatrix}1\\0\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_1 + x_3 - x_4 = 0$$

$$\vec{b}^\mathsf{T}\vec{x} = \vec{b}\cdot\vec{x} = 0 \quad \ldots \quad \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_2 - x_3 + x_4 = 0$$

$$\underbrace{\left[\begin{array}{cccc|c}1 & 0 & 1 & -1 & 0\\0 & 1 & -1 & 1 & 0\end{array}\right]}_{\left[\begin{array}{c|c}\vec{a}^\mathsf{T} & 0\\ \vec{b}^\mathsf{T} & 0\end{array}\right]}$$

$$x_1 = x_4 - x_3$$
$$x_2 = x_3 - x_4$$

$$U \ni \vec{x} = \begin{bmatrix}x_4 - x_3\\x_3 - x_4\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}-1\\1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}, \quad B_U = \left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right\}, \quad \dim U = 2.$$

(b) Ena možnost za $B = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$, $N(B) = U$.

Druga možnost: $B_1 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & -2 \end{bmatrix}$, $N(B_1) = U$.

Ena možnost za $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C(A) = U$.

Druga možnost: $A_1 = \begin{bmatrix} -1 & 1 & 0 & -2 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$, $C(A_1) = U$

(c) Ali so kakšne ovire, da bi taka F lahko obstajala?

$$\dim(N(F)) + \dim(C(F)) = \dim \mathbb{R}^m = \dim \mathbb{R}^4 = 4$$

$\dim(N(F)) = \dim(U) = 2$, $\dim(C(F)) = \dim(U) = 2$, m = št. stolpcev F

... Dimenzijskih ovir ni.

Poskusimo poiskati F:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} -1 & 1 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

... $C(A) = C(F)$

$N(B) = N(F)$, $C(A) = N(B) = U$

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$F = AB$.

4. Dani so matrika K ter vektorja **a** in **b**:

$$K = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 0 & -1 & 1 \\ 2 & 1 & -3 & 2 \\ 2 & 1 & -3 & 2 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Ali sta vektorja **a** in **b** vsebovana v $N(K)$? ... v $C(K)$?

(b) Poišči baze in določi dimenzije podprostorov $N(K)$, $C(K)$, $N(K) \cap C(K)$ ter $N(K) + C(K)$.

(a) Po def.: $\vec{x} \in N(K)$, če je $K\vec{x} = \vec{0}$. Preverimo za $\vec{a}$ in $\vec{b}$:

$$K\vec{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \quad \text{in} \quad \vec{a} \in N(K). \quad K\vec{b} = \begin{bmatrix} 0 \\ 1 \\ * \\ * \end{bmatrix} \neq \vec{0} \ldots \vec{b} \notin N(K).$$

Po def.: $\vec{y} \in C(K)$, če je $\vec{y} = K\vec{x}$ za nek $\vec{x}$ oz. sistem $K\vec{x} = \vec{y}$ ima rešitev. To preverimo z G.e.

$$[K \,|\, \vec{a}, \vec{b}] = \left[\begin{array}{cccc|cc} 1 & 1 & -2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 2 & 1 & -3 & 2 & 1 & 0 \\ 2 & 1 & -3 & 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cc} 1 & 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right] \longrightarrow$$

$$\longrightarrow \left[\begin{array}{cccc|cc} x_1 & x_2 & x_3 & x_4 & & \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$K\vec{x} = \vec{a}$ ni protisloven ... ima rešitve ... $\vec{a} \in C(K)$

$K\vec{x} = \vec{b}$ ni protisloven ... $\vec{b} \in C(K)$.

(b) Baza za $C(K)$: $B_{C(K)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ (pivotna stolpca originalne K)

$\vec{c}_1 \quad \vec{c}_2$ $\quad \therefore \dim(C(K)) = 2$

Baza za $N(K)$: $\begin{array}{l} x_1 - x_3 + x_4 = 0 \\ x_2 - x_3 = 0 \end{array} \ldots \begin{array}{l} x_1 = x_3 - x_4 \\ x_2 = x_3 \end{array}$ $\quad \dim(N(K)) = 2 \quad \therefore$

$$N(K) \ni \vec{x} = \begin{bmatrix} x_3 - x_4 \\ x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ldots B_{N(K)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\vec{n}_1 \quad \vec{n}_2$

$$N(K) \cap C(K) = \{ \vec{x} \in \mathbb{R}^4 : \underbrace{\vec{x} \in N(K)}_{\vec{x} = \alpha_1 \vec{n}_1 + \alpha_2 \vec{n}_2} \text{ in } \underbrace{\vec{x} \in C(K)}_{\vec{x} = \beta_1 \vec{c}_1 + \beta_2 \vec{c}_2} \}$$

Kdaj je lahko $\alpha_1 \vec{n}_1 + \alpha_2 \vec{n}_2 = \beta_1 \vec{c}_1 + \beta_2 \vec{c}_2$?

$$\underbrace{\alpha_1 \vec{n}_1 + \alpha_2 \vec{n}_2 - \beta_1 \vec{c}_1 - \beta_2 \vec{c}_2}_{= \left[\vec{n}_1, \vec{n}_2, \vec{c}_1, \vec{c}_2 \right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ -\beta_1 \\ -\beta_2 \end{bmatrix}} = \vec{0}$$

Naredimo G.e. na: $\longrightarrow \left[\vec{n}_1, \vec{n}_2, \vec{c}_1, \vec{c}_2 \right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ -\beta_1 \\ -\beta_2 \end{bmatrix}$

$$\left[\vec{n}_1, \vec{n}_2, \vec{c}_1, \vec{c}_2 \right] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Kaj je $N(K) + C(K)$? $N(K) + C(K) = \{ \vec{x} + \vec{y} \,,\, \vec{x} \in N(K), \vec{y} \in C(K) \} =$

$$= \{ \alpha_1 \vec{n}_1 + \alpha_2 \vec{n}_2 + \underset{-\beta_1}{\beta_1'} \vec{c}_1 + \underset{-\beta_2}{\beta_2'} \vec{c}_2 : \alpha_1, \alpha_2, \beta_1', \beta_2' \in \mathbb{R} \}$$

$$B_{N(K)+C(K)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\} , \quad B_{N(K) \cap C(K)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\underbrace{\dim (N(K) + C(K))}_{= 3} + \underbrace{\dim (N(K) \cap C(K))}_{= 1} = \underbrace{\dim \mathbb{R}^4}_{= 4}$$