Schur Form and a Proof of Existence

Statement of Schur's Theorem

Let $A \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^* AQ = T,$$

where T is upper triangular.

This factorization is called the *Schur decomposition*, and T is called a *Schur form* of A.

Because Q is unitary, T is unitarily similar to A, so A and T have the same eigenvalues. Since T is upper triangular, its eigenvalues are exactly its diagonal entries. Thus the diagonal of T consists of the eigenvalues of A, in some order.

Why Schur Form Matters

Schur form is important because it says that every complex square matrix can be reduced, by a unitary change of basis, to an upper triangular matrix. This is stronger than merely knowing that eigenvalues exist: it organizes the whole matrix into a structured form while preserving numerical stability through a unitary transformation.

In the QR algorithm, Schur form is the natural target: repeated QR steps aim to drive a matrix toward an upper triangular matrix (or, in the real case, a quasi-upper-triangular one).

Proof of Existence

We prove Schur's theorem by induction on n.

Theorem 1 (Schur)**.** *For every $A \in \mathbb{C}^{n\times n}$, there exists a unitary matrix Q such that Q^*AQ is upper triangular.*

Proof. We proceed by induction on n.

Base case: $n = 1$. A 1×1 matrix is already upper triangular, so the claim is immediate.

Inductive step. Assume the theorem holds for all complex matrices of size $(n-1) \times (n-1)$, and let $A \in \mathbb{C}^{n\times n}$.

Since $\mathbb{C}$ is algebraically closed, the characteristic polynomial of A has a root. Therefore A has at least one eigenvalue $\lambda \in \mathbb{C}$, with corresponding nonzero eigenvector v:

$$Av = \lambda v.$$

Normalize v by setting

$$q_1 = \frac{v}{\|v\|}.$$

Then q_1 is a unit vector. Extend q_1 to an orthonormal basis of $\mathbb{C}^n$:

$$\{q_1, q_2, \ldots, q_n\}.$$

Let

$$Q_1 = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}.$$

By construction, Q_1 is unitary.

Now consider

$$Q_1^* A Q_1.$$

Its first column is

$$Q_1^* A q_1 = Q_1^*(\lambda q_1) = \lambda Q_1^* q_1 = \lambda e_1,$$

where $e_1 = (1, 0, \ldots, 0)^T$. Hence $Q_1^* A Q_1$ has the block form

$$Q_1^* A Q_1 = \begin{bmatrix} \lambda & * \\ 0 & B \end{bmatrix},$$

where $B \in \mathbb{C}^{(n-1)\times(n-1)}$.

By the induction hypothesis, there exists a unitary matrix $U \in \mathbb{C}^{(n-1)\times(n-1)}$ such that

$$U^* B U = T_2,$$

with T_2 upper triangular.

Define

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}.$$

Then Q_2 is unitary, and

$$Q_2^* \begin{bmatrix} \lambda & * \\ 0 & B \end{bmatrix} Q_2 = \begin{bmatrix} \lambda & * \\ 0 & U^* B U \end{bmatrix} = \begin{bmatrix} \lambda & * \\ 0 & T_2 \end{bmatrix}.$$

This matrix is upper triangular.

Finally, set

$$Q = Q_1 Q_2.$$

Since Q_1 and Q_2 are unitary, Q is unitary. Moreover,

$$Q^* A Q = Q_2^* Q_1^* A Q_1 Q_2,$$

which is upper triangular by the previous computation.

Therefore there exists a unitary Q such that $Q^* A Q$ is upper triangular. □

Interpretation

The proof works by choosing one eigenvector, placing it as the first basis vector, and thereby forcing the first column of the transformed matrix to have zeros below the diagonal. One then repeats the process on the remaining $(n-1) \times (n-1)$ block.

So Schur form can be viewed as a repeated extraction of eigenvectors, organized through orthonormal bases and unitary similarity transformations.

Real Schur Form

If $A \in \mathbb{R}^{n\times n}$, an upper triangular reduction over $\mathbb{R}$ need not exist, because A may have complex eigenvalues. However, there is still a *real Schur form*: there exists an orthogonal matrix Q such that

$$Q^T A Q = R,$$

where R is block upper triangular with diagonal blocks of size 1×1 and 2×2. The 1×1 blocks correspond to real eigenvalues, and the 2×2 blocks correspond to complex-conjugate pairs.