Orthogonal polynomials

Thm: Let $g_0, \dots g_n$ be a sequence of orthogonal polynomials with respect to an inner product $\langle \cdot, \cdot \rangle$, then $\langle g_k, p \rangle = 0$ for p with degree at most $k-1$

Pf: $p(x) = \sum_{i=0}^{k-1} c_i g_i(x)$ so the result is immediate

Thm: There is a 3 term recursive relation for orthogonal polynomials

$$g_{i+1}(x) = (x - \alpha_{i+1}) g_i(x) - \beta_i g_{i-1}(x)$$

w/ $\alpha_{i+1} = \dfrac{\langle x g_i, g_i \rangle}{\langle g_i, g_i \rangle}$ $\qquad \beta_0 = 0$

$$\beta_i = \frac{\langle x g_i, g_{i-1} \rangle}{\langle g_{i-1}, g_{i-1} \rangle}$$

see sheet.

Gaussian Quadrature

Let $g_0, \dots, g_n$ be a sequence of orthogonal polynomials & define the inner product

$$\langle f, g \rangle_w = \int_a^b f \cdot g \cdot w \, dx$$

where $w > 0$. Then g_i has i different zeros on $[a,b]$

Pf Let $x_1 \dots x_k$ be zeroes of g_i if $k < i$

$$0 = \langle \underbrace{(x-x_1) \dots (x-x_k)}_{q}, g_i \rangle = \int_a^b q g_i w > 0$$

Contradiction.

Note we chose f to induce a contradiction.

Gaussian Quadrature of order n

- allows us to compute / evaluate the integral

$$\int_a^b f(x) w(x) \, dx \approx \sum_{i=1}^{n} q_i f(x_i)$$

x_i - quadrature nodes (zeros of orthogonal polynomial for w)

q_i - weights

$\Rightarrow$ is exact for polynomials of degree $< 2n-1$

Numerical Integration

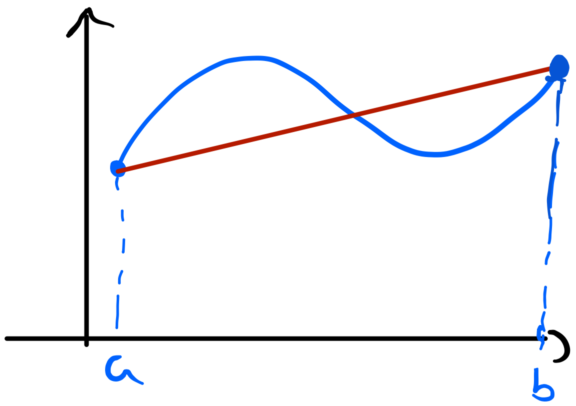
Goal: evaluate $\int_a^b f(x)dx = F(b) - F(a)$

if we cannot evaluate the indefinite integral

$$e^{-x^2}, \quad \frac{\sin x}{x} \quad x\tan x$$

Basic: Trapezoidal rule

$\int_a^b f(x)dx$ do a linear approximation



$$p(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

if a ; b are close $\int_a^b f(x)dx \approx \int_a^b p(x)dx$

$$= f(a)(b-a) + \frac{f(b)-f(a)}{b-a} \frac{(b-a)^2}{2}$$

$$= \boxed{\frac{b-a}{2}(f(a)+f(b))}$$

Error

a & b are close

$$E = \int_a^b (f(x) - p(x))\,dx$$

$$\int_a^b (f(x)-p(x))\,dx = \int_a^b f[a,b,x]\,(x-b)(x-a)\,dx$$

$$= f[a,b,\xi] \int_a^b (x-b)(x-a)\,dx$$

mean value thm

$$= \frac{f''(\eta)}{2}\left(-\frac{1}{3}(b-a)^3\right)$$

see newton remainder thm

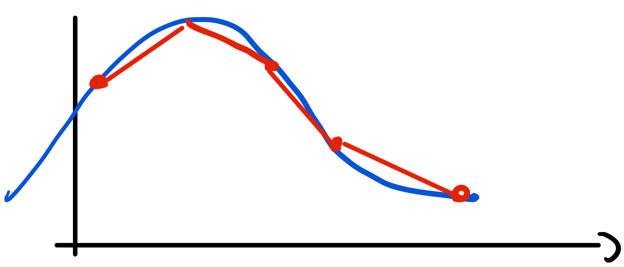
$$= -\frac{(b-a)^3}{12} f''(\eta)$$

$\eta, \xi \in [a,b]$ exist by MVT

If the interval is not short → divide $[a,b]$ into pieces w/ equidistant points $x_0 \dots x_n$ with $h = h_i = x_{i+1} - x_i$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{i=0}^{n-1} f(x_i) + f(x_{i+1}) =$$

$$= \frac{h}{2}\left(f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\right)$$



$E_i \Rightarrow$ error on each interval

$$E_i = -\frac{h^3 f''(\eta_i)}{12} \qquad \eta\ (x_i, x_{i+1})$$

$$E = \sum_{i=0}^{n-1} E_i = -\frac{h^3 f''(\eta_i)}{12} = -n\frac{h^3 f''(\eta)}{12}$$

$$= -\frac{(b-a)\cdot h^2 f''(\eta)}{12}$$

$\eta \in [a,b]$

Example

$$\int_0^1 e^{-x^2}dx \quad z \text{ napako } 10^{-6}$$

$$\left|\frac{(b-a)h^2 f''(\eta)}{12}\right| < 10^{-6}$$

$f'(x) = 2xe^{-x^2}$ $\quad f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2}$

$$= -2e^{-x^2}(1-2x^2)$$

$f'''(x) > 0$ on $[0,1]$

so f'' is increasing on $[0,1]$

but $f''(x)$ crosses zero on $[0,1]$

at $f''(0) = -2$ $\quad f''(1) = \frac{2}{e}$

$|f''(0)| \geq |f''(1)|$

$\uparrow$

so we take this

$$\frac{|b-a|\, 2h^2}{12} < 10^{-6}$$

$h^2 \leq 6 \cdot 10^{-6}$

$h = \sqrt{1/6} \cdot 10^3$

≈ 410

Tai's Model - example of reinvention

Trapazoid rule w/ control step

set h beforehand

to estimate the error → we need to estimate / bound $f''(\eta)$ on $[a,b]$

Let $I = \int_a^b f(x)\,dx$ $\quad T(h)$ estimate for I

$$E(h) := T(h) - I = \frac{b-a}{12} f''(\xi_h) h^2$$

$\xi_h \in (a,b)$ $\quad$ ↑ still need this

$$E(h/2) := T(h/2) - I = \frac{b-a}{12} f''(\xi_{h/2}) \frac{h^2}{2}$$

$\xi_{h/2} \in (a,b)$

- Assume $\frac{b-a}{12} f''(\xi_h)$ is a constant at each step $\forall h$

$$I = T(h) - Ch^2 = T(h/2) - C\frac{h^2}{4}$$

$$T(h) - T(h/2) = \frac{3}{4}Ch^2 + O(h^4)$$

$$\Rightarrow Ch^2 \approx 4/3\,(T(h) - T(h/2))$$

so $\underbrace{\frac{4}{3}(T(h) - T(h/2))}_{\text{estimate for error at } h}$; $\underbrace{1/3\,(T(h) - T(h/2))}_{\text{estimate for error at } h/2}$

$E(h)$

$E(h/2)$

Alg

① $T(b-a) = (b-a)\frac{f(a)+f(b)}{2} \Rightarrow T(h)$

② $T((b-a)/2) = \frac{T(b-a)}{2} + \frac{b-a}{2} f\left(\frac{a+b}{2}\right) \Rightarrow T(h/2)$

③ $1/3\,(T(b-a) - T((b-a)/2))$

↳ if this is small enough in abs. value then stop

Above implies

$$T(h/2) = \frac{T(h)}{2} + \underbrace{\frac{h}{2}\sum_{i=1}^{n} f(a + (i - 1/2)h)}$$

half $T(h)$

$n = \frac{b-a}{h}$

n h

just comp. this part

This follows from

h, $2h$, $3h$, ...

$\Rightarrow$ in difference only the midpoints remain

Adaptive trapazoid rule

w/ the above method h doesn't have to be determined

$\Rightarrow$ but what if we wanted it adaptive.

Recursive comp.

$T(b-a)$ & $T((b-a)/2)$

error estimate

$$e := \frac{T(b-a)/2 - T(b-a)}{3}$$

if e is small enough $\Rightarrow$ stop & return $T((b-a)/2) + e$

if e is too big

$\Rightarrow$ apply above to $[a, (a+b)/2]$ $[(a+b)/2, b]$

error $\Rightarrow$ half of starting tolerance

$\downarrow$

repeat recursively

See additional writeup