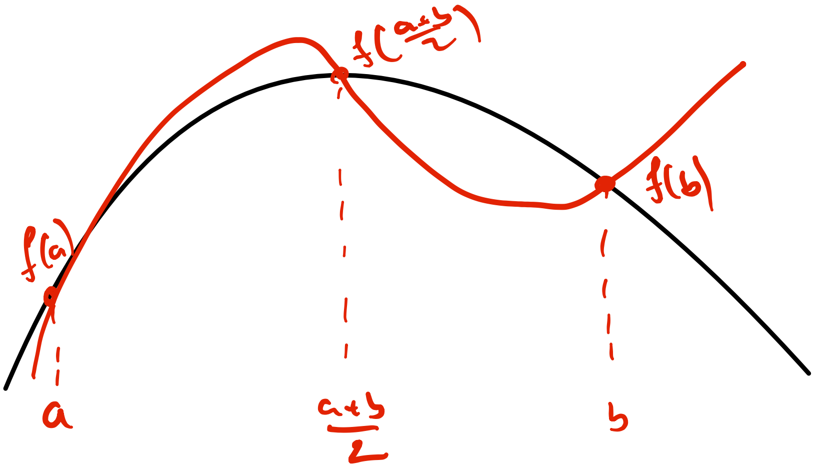
Simpson's Rule

p_2 : polynomial of degree 2 which interpolates

$$(a, f(a)),\ \left(\tfrac{a+b}{2}, f\left(\tfrac{a+b}{2}\right)\right),\ (b, f(b))$$



$$\boxed{p_2(x) = c_0 + c_1(x-a) + c_2(x-a)\left(x - \tfrac{a+b}{2}\right)}$$

Let $h = \frac{b-a}{2}$ Solve system

$$p_2(a) = f(a)$$
$$p_2\left(\tfrac{a+b}{2}\right) = f\left(\tfrac{a+b}{2}\right)$$
$$p_2(b) = f(b)$$

$$c_0 = f[a] = f(a)$$

$$c_1 = f\left[a, \tfrac{a+b}{2}\right] = \frac{f(a+h) - f(a)}{h}$$

$$c_2 = f\left[a, \tfrac{a+b}{2}, b\right] = \frac{f(a+2h) - 2f(a+h) + f(a)}{2h^2}$$

Compute $\int_a^b p_2(x)\,dx$ & use $x = a+t$

$$\Rightarrow \int_a^{a+2h} p_2(x)\,dx = \int_0^{2h} p_2(a+t)\,dt = f(a)\cdot 2h + \frac{f(a+h)-f(a)}{h}\cdot 2h^2$$

$$+ \frac{f(a+2h)-2f(a+h)+f(a)}{h^2}\cdot 3h^3$$

$$\boxed{= \frac{h}{3}\left(f(a) + 4f(a+h) + f(a+2h)\right)}$$

error on the order of $-\frac{1}{90} h^5 f^{(4)}(\xi)$ $\xi \in [a,b]$

Composite Simpson's rule

Equidistant partition $P = \{x_0 = a < x_1 < \ldots < x_n = b\}$

$h = x_{i+1} - x_i$

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n/2-1} \frac{h}{3}\left(f(x_{2i}) - 4f(x_{2i+1}) - f(x_{2i+2})\right)$$

$$= \frac{h}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + f(x_n)\Big]$$

Each error $E_i = -\dfrac{h^5 f^{(4)}(\eta_i)}{90}$

$$\sum_{i=0}^{n/2-1} E_i = \sum_{i=0}^{n/2-1} -\frac{h^5 f^{(4)}(\eta_i)}{90} = -\frac{n}{2}\,\frac{h^5 f^{(4)}(\eta)}{90}$$

$$= -\frac{(b-a)}{80}\, h^4 f^{(4)}(\eta)$$

Adaptive Simpson's Rule

→ same idea as before

$S(h)$: simpson's rule w/ value h

① Compute $S(b-a)$ & $S((b-a)/2)$

② $$\int_a^b f(x)dx \approx S(h) + C_1(h^4) = S(h/2) + C_1\left(\frac{h}{2}\right)^4$$

$$C_1\left(\frac{h}{2}\right)^4 = \frac{S((b-a)/2) - S(b-a)}{15} = E$$
(error estimate)

③ if E is too large, recurse on each side.

Newton-Coates Rule

$[a,b]$ is divided into $n+1$ equidistant parts w/ estimates : $\int_a^b p_n(x)dx$ is the polynomial approximation

name	n	formula
trapezoid	1	$\frac{b-a}{2}[f(a) + f(b)]$
simp 1/3	2	$\frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$
simp 3/8	3	$\frac{b-a}{8}[f(a) + 3f(a+h) \ 3f(b-h) + f(b)]$
boolovo	4	$\frac{b-a}{90}[7f(a) + 32f(a+h) + 12f(\frac{a+b}{2}) + 32f(b-h) + f(b)]$

ime	n	Error
trapezoid	1	h^2
simp $1/3$	2	h^4
simp $3/8$	3	h^4
boolovo	4	h^6

Derivation of Newton-Coates rule

We want to derive an integration formula on $n+1$ equally space points $x_0 = a < x_1 < \dots < x_n = b$ with spacing $h = \frac{b-a}{n}$

① build interpolation polynomials

② approximate

$$\int_a^{a+nh} f(x)\,dx = \underbrace{\sum_{i=0}^{n} a_i f(a+ih)}_{\text{assume this form}} + R(f(x))$$

we are looking for a_i's & $R(f(x))$ is the error

For Simp. 3/8 rule

$$\int_0^{3h} f(x)\,dx = a_0 f(0) + a_1 f(h) + a_2 f(2h) + a_3 f(3h)$$

w/ $R(1) = R(x) = R(x^2) = R(x^3) = 0$

$\rightarrow$ should be exact for all polynomials of degree at most 3

$$\int_0^{3h} 1\,dx = 3h = a_0 + a_1 + a_2 + a_3$$

$$\int_0^{3h} x\,dx = \frac{9}{2} h^2 = a_0 \cdot 0 + a_1 \cdot h + a_2 \cdot 2h + a_3 \cdot 3h$$

$$\int_0^{3h} x^2\,dx = \frac{27}{3} h^3 = a_0 \cdot 0 + a_1 h^2 + a_2\, 4h^2 + a_3\, 9h^2$$

$$\int_0^{3h} x^3\,dx = \frac{81}{4} h^4 = a_0 \cdot 0 + a_1 h^3 + a_2\, 8h^3 + a_3\, 27h^3$$

$$\Rightarrow a_0 = \tfrac{3}{8}h \qquad a_1 = \tfrac{9}{8}h \qquad a_2 = \tfrac{9}{8}h \qquad a_3 = \tfrac{3}{8}h$$

We expect an error like

$$D \cdot f^{(4)}(\xi) \qquad \xi \in [a, b] \quad \text{for } f(x) = x^4$$

$$\int_0^{3h} x^4\,dx = \frac{3^5}{5} h^5 = \tfrac{3}{8} h\left(3h^4 + 3\cdot 2^4 h^4 + 3^4 \cdot h^4\right) + \underbrace{24D}_{\substack{\text{by assumption} \\ f^{(4)}(x^4) = 24}}$$

$$D = -\tfrac{3}{80} h^5$$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{b-a}{8}\left[f(a) + 3f(a+h) + 3f(b-h) + f(b)\right] - \frac{(b-a)}{80} h^5 f^{(4)}(\xi)$$

Romberg's method

Let $T(h)$ be the approximation of $I = \int_a^b f(x)\,dx$ w/ the trapazoid rule. w/ step size h.

If our function is $(2k+2)$ differentiable we can write the error by a convergent series

$$E(h) = I - T(h) = c_1 h^2 + c_2 h^4 + \cdots + c_k h^{2k} + O(h^{2k+2})$$

c_i's do not depend on h

Compute at $h, h/2, h/4 \ \ldots \ h/2^k$

$$I = T(h) + c_1 h^2 + \cdots + c_k h^{2k} + O(h^{2k+2})$$

$$I = T(h/2) + c_1\left(\frac{h}{2}\right)^2 + \cdots + c_k\left(\frac{h}{2}\right)^{2k} + \cdots$$

$\times 4$

$$4I = 4T(h/2) + c_1 h^2 + 4c_2\left(\frac{h}{2}\right)^4 + \cdots$$

$\downarrow$

$$3I = 4T(h/2) - T(h) + c_2\left(\frac{1}{2^2} - 1\right)h^4 + \cdots$$

$$I = \underbrace{\frac{4T(h/2) - T(h)}{3}}_{T_1(h/2) \text{ definition}} + \frac{c_2}{3}\left(\frac{1}{2^2} - 1\right)h^4 + \cdots$$

Similarly

$$I = \underbrace{\frac{4T(h/4) - T(h/2)}{3}}_{T_1(h/4)} + \dots$$

Generally

$$I = \frac{4T(\frac{h}{2^k}) - T(h/2^{k-1})}{3} + C_2/3 \left(\frac{1}{2^2} - 1\right)\left(\frac{h}{2^{k-1}}\right)^4$$

$$+ \dots$$

$$\Rightarrow T_1(h/2^l) = \frac{4T(h/2^l) - T(h/2^{l-1})}{3}$$

$$\Rightarrow I = T_1(h/2^l) + O(h^4)$$

can continue

$$T_2(h/2^l) = \frac{4^2 T_1(h/2^l) - T_1(h/2^{l-1})}{4^2-1}$$

$$l = 2, 3, \dots$$

$$T_m(h/2^l) = \frac{4^m T_{m-1}(h/2^l) - T_{m-1}(h/2^{l-1})}{4^m - 1}$$

Table

$T(n)$

$T(n/2)$ $\quad T_1(n/2)$

$T(n/2^2)$ $\quad T_1(n/2^2)$ $\quad T_2(n/2^2)$

$T(n/2^3)$ $\quad T_1(n/2^3)$ $\quad T_2(n/2^3)$ $\quad T_3(n/2^3)$

To get $T_m(n/2^e)$

1. multiply element to left by $\frac{4^m}{4^m-1}$

2. subtract one up & one left w/ weight $\frac{1}{4^m-1}$