Derivation of Newton's Interpolation Remainder Theorem

Introduction

Let

$$x_0, x_1, \ldots, x_n$$

be distinct interpolation nodes, and let

$$p_n(x)$$

be the unique polynomial of degree at most n satisfying

$$p_n(x_i) = f(x_i), \qquad i = 0, \ldots, n.$$

We define the interpolation error by

$$R_n(x) = f(x) - p_n(x).$$

Our goal is to derive an explicit formula for this remainder term.

Newton Interpolation Formula

The Newton form of the interpolating polynomial is

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i),$$

where

$$f[x_0, \ldots, x_k]$$

denotes the divided difference of order k.

Now introduce one additional point x. The interpolation polynomial through the points

$$x_0, x_1, \ldots, x_n, x$$

has degree $n + 1$, and its Newton form is

$$p_{n+1}(t) = p_n(t) + f[x_0, \ldots, x_n, x] \prod_{i=0}^{n} (t - x_i).$$

Evaluation at $t = x$

Since p_{n+1} interpolates the function f also at the point x, we have

$$p_{n+1}(x) = f(x).$$

Substituting $t = x$ into the previous expression gives

$$f(x) = p_n(x) + f[x_0, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i).$$

Rearranging yields

$$\boxed{f(x) - p_n(x) = f[x_0, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i)}.$$

This is the **Newton divided-difference remainder formula**.

Derivative Form of the Remainder

If the function f possesses $n + 1$ continuous derivatives on an interval containing the interpolation nodes and the point x, then the divided difference satisfies

$$f[x_0, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some

$$\xi$$

lying between the interpolation nodes and x.

Substituting this into the previous formula gives

$$\boxed{f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)}.$$

This is the classical interpolation remainder theorem.

Interpretation

The factor

$$\prod_{i=0}^{n} (x - x_i)$$

ensures that the error vanishes at every interpolation node:

$$R_n(x_i) = 0, \qquad i = 0, \ldots, n.$$

The remaining coefficient depends on the $(n+1)$-st derivative of the function, showing that interpolation accuracy is controlled by the first derivative not captured exactly by the interpolating polynomial.

Example: Linear Interpolation

Suppose we interpolate at two points

$$a, b.$$

The interpolation polynomial is linear:

$$p_1(x).$$

The remainder formula becomes

$$f(x) - p_1(x) = f[a, b, x](x - a)(x - b).$$

Since

$$f[a, b, x] = \frac{f''(\xi)}{2},$$

we obtain

$$\boxed{f(x) - p_1(x) = \frac{f''(\xi)}{2}(x - a)(x - b)}.$$

This formula is the basis for the error analysis of the trapezoidal rule.