The Three-Term Recurrence for Orthogonal Polynomials

1 Setup

Let $I \subseteq \mathbb{R}$ be an interval and let $w(x) \geq 0$ be a weight function on I such that all moments exist. Define the inner product

$$\langle f, g \rangle = \int_I f(x)g(x)w(x)\,dx.$$

A sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ is called an *orthogonal polynomial sequence* if

$$\deg p_n = n$$

and

$$\langle p_n, p_m \rangle = 0 \qquad \text{whenever } n \neq m.$$

In this note we assume that each p_n is *monic*, meaning that the leading coefficient of p_n is 1.

2 Statement of the Three-Term Recurrence

Theorem 1 (Three-term recurrence). *Let $\{p_n\}_{n=0}^{\infty}$ be a monic orthogonal polynomial sequence. Then there exist real numbers α_n and positive numbers β_n such that*

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad n \geq 1.$$

Equivalently,

$$xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x).$$

The coefficients are

$$\alpha_n = \frac{\langle xp_n, p_n \rangle}{\langle p_n, p_n \rangle}, \qquad \beta_n = \frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle}.$$

3 Proof

Proof. Since p_n is monic of degree n, the polynomial $xp_n(x)$ has degree $n+1$ and leading coefficient 1. Because the polynomials $p_0, p_1, \ldots, p_{n+1}$ form a basis for all polynomials of degree at most $n+1$, we may write

$$xp_n(x) = p_{n+1}(x) + c_n p_n(x) + c_{n-1}p_{n-1}(x) + \cdots + c_0 p_0(x).$$

The coefficient of p_{n+1} is 1 because both xp_n and p_{n+1} are monic of degree $n+1$.

We now show that all coefficients below p_{n-1} vanish. Let $k \leq n-2$. Taking the inner product with p_k gives

$$\langle xp_n, p_k \rangle = \langle p_n, xp_k \rangle.$$

But xp_k has degree at most $k+1 \le n-1$. Hence xp_k lies in the span of $p_0, p_1, \ldots, p_{n-1}$. Since p_n is orthogonal to every polynomial of degree at most $n-1$, we get

$$\langle p_n, xp_k \rangle = 0.$$

Therefore

$$\langle xp_n, p_k \rangle = 0$$

for all $k \le n-2$. By orthogonality, this implies

$$c_k = 0 \qquad \text{for } k = 0, 1, \ldots, n-2.$$

Thus only three terms remain:

$$xp_n(x) = p_{n+1}(x) + c_n p_n(x) + c_{n-1} p_{n-1}(x).$$

Set

$$\alpha_n = c_n, \qquad \beta_n = c_{n-1}.$$

Taking the inner product with p_n gives

$$\langle xp_n, p_n \rangle = \alpha_n \langle p_n, p_n \rangle,$$

so

$$\alpha_n = \frac{\langle xp_n, p_n \rangle}{\langle p_n, p_n \rangle}.$$

Taking the inner product with p_{n-1} gives

$$\langle xp_n, p_{n-1} \rangle = \beta_n \langle p_{n-1}, p_{n-1} \rangle.$$

By symmetry of the inner product,

$$\langle xp_n, p_{n-1} \rangle = \langle p_n, xp_{n-1} \rangle.$$

Using the same recurrence one degree lower, the leading part of xp_{n-1} is p_n, and all other terms are orthogonal to p_n. Therefore

$$\langle p_n, xp_{n-1} \rangle = \langle p_n, p_n \rangle.$$

Hence

$$\beta_n = \frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle}.$$

Since $\langle p_n, p_n \rangle > 0$, it follows that $\beta_n > 0$. □

4 Example: Monic Legendre Polynomials

Example 1. *Consider the interval* $[-1, 1]$ *with weight* $w(x) = 1$*. The inner product is*

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx.$$

The first few monic Legendre polynomials are

$$p_0(x) = 1,$$

$$p_1(x) = x,$$

$$p_2(x) = x^2 - \frac{1}{3},$$
$$p_3(x) = x^3 - \frac{3}{5}x.$$

They are the usual Legendre polynomials scaled so that the leading coefficient is 1.

Because the weight is even on the symmetric interval $[-1, 1]$*, the coefficients* α_n *vanish:*

$$\alpha_n = 0.$$

Thus the recurrence has the form

$$p_{n+1}(x) = xp_n(x) - \beta_n p_{n-1}(x).$$

Let us verify this for $n = 1$. *We have*

$$xp_1(x) = x^2.$$

Since

$$p_2(x) = x^2 - \frac{1}{3},$$

we obtain

$$xp_1(x) = p_2(x) + \frac{1}{3}p_0(x).$$

Equivalently,

$$p_2(x) = xp_1(x) - \frac{1}{3}p_0(x).$$

So

$$\beta_1 = \frac{1}{3}.$$

Now verify the next step. Since

$$xp_2(x) = x^3 - \frac{1}{3}x$$

and

$$p_3(x) = x^3 - \frac{3}{5}x,$$

we get

$$xp_2(x) - p_3(x) = \left(-\frac{1}{3} + \frac{3}{5}\right)x = \frac{4}{15}x.$$

Hence

$$xp_2(x) = p_3(x) + \frac{4}{15}p_1(x),$$

or equivalently,

$$p_3(x) = xp_2(x) - \frac{4}{15}p_1(x).$$

Thus

$$\beta_2 = \frac{4}{15}.$$

5 Interpretation

The essential reason only three terms appear is that multiplication by x raises the degree by one, while orthogonality eliminates all components of degree $n - 2$ and below. Therefore xp_n can only have components in the directions

$$p_{n+1}, \qquad p_n, \qquad p_{n-1}.$$

This is the structure that makes orthogonal polynomials especially useful in approximation theory, Gaussian quadrature, and spectral methods.