Derivation of Simpson's Rule

1 Goal

We want to approximate a definite integral

$$I = \int_a^b f(x)\,dx$$

by replacing the function f with a simple interpolating polynomial.

Simpson's rule is obtained by approximating f on an interval using a quadratic polynomial through three equally spaced points.

2 Setup

Let

$$m = \frac{a+b}{2},$$

so that m is the midpoint of the interval $[a, b]$. Define

$$h = \frac{b-a}{2}.$$

Thus,

$$a = m - h, \qquad b = m + h.$$

We approximate $f(x)$ by the quadratic polynomial $p_2(x)$ interpolating f at

$$x = a, \qquad x = m, \qquad x = b.$$

The approximation is then

$$\int_a^b f(x)\,dx \approx \int_a^b p_2(x)\,dx.$$

3 Quadratic Interpolation

It is convenient to shift coordinates by writing

$$x = m + ht,$$

where

$$t \in [-1, 1].$$

The three interpolation points become

$$t = -1, \qquad t = 0, \qquad t = 1.$$

Let

$$F(t) = f(m + ht).$$

The quadratic interpolant through $F(-1)$, $F(0)$, and $F(1)$ is

$$P_2(t) = F(-1)\frac{t(t-1)}{2} + F(0)(1-t^2) + F(1)\frac{t(t+1)}{2}.$$

Since $dx = h\,dt$, we have

$$\int_a^b f(x)\,dx = h\int_{-1}^1 F(t)\,dt \approx h\int_{-1}^1 P_2(t)\,dt.$$

4 Integrating the Basis Polynomials

Compute each contribution:

$$\int_{-1}^1 \frac{t(t-1)}{2}\,dt = \frac{1}{2}\int_{-1}^1 (t^2 - t)\,dt = \frac{1}{3},$$

because the integral of the odd function t over $[-1, 1]$ is zero.

Similarly,

$$\int_{-1}^1 (1-t^2)\,dt = 2 - \frac{2}{3} = \frac{4}{3},$$

and

$$\int_{-1}^1 \frac{t(t+1)}{2}\,dt = \frac{1}{3}.$$

Therefore,

$$\int_{-1}^1 P_2(t)\,dt = \frac{1}{3}F(-1) + \frac{4}{3}F(0) + \frac{1}{3}F(1).$$

Returning to x-coordinates,

$$F(-1) = f(a), \qquad F(0) = f(m), \qquad F(1) = f(b).$$

Hence,

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\big[f(a) + 4f(m) + f(b)\big].$$

Since $h = (b-a)/2$, this can also be written as

$$\boxed{\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].}$$

This is Simpson's rule.

5 Why the Weights Are $1, 4, 1$

The weights $1, 4, 1$ arise from integrating the three Lagrange basis polynomials. The midpoint receives the largest weight because its basis polynomial

$$1 - t^2$$

has the largest area over $[-1, 1]$.

6 Exactness

Although Simpson's rule is derived using quadratic interpolation, it is exact for all polynomials of degree at most 3.

That is, if f is any cubic polynomial, then

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

This extra degree of exactness occurs because the cubic error term is odd about the midpoint and integrates to zero on the symmetric interval.

7 Error Term

If $f \in C^4([a,b])$, then the error in Simpson's rule is

$$\int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

for some $\xi \in (a,b)$.

Thus, the error is controlled by the fourth derivative of f.

8 Composite Simpson's Rule

To approximate

$$\int_a^b f(x)\,dx,$$

divide the interval into n equal subintervals, where n must be even.

Let

$$h = \frac{b-a}{n}, \qquad x_i = a + ih.$$

Then the composite Simpson's rule is

$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1\\ i \text{ odd}}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i \text{ even}}}^{n-2} f(x_i) + f(x_n)\right]}.$$

The weights therefore follow the pattern

$$1,\ 4,\ 2,\ 4,\ 2,\ \ldots,\ 4,\ 1.$$

9 Example

Approximate

$$I = \int_0^1 e^x\,dx.$$

Here,

$$a = 0, \qquad b = 1, \qquad m = \frac{1}{2}.$$

Using Simpson's rule,

$$S = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right].$$

Since $f(x) = e^x$,

$$S = \frac{1}{6}\left[1 + 4e^{1/2} + e\right].$$

The exact value is

$$I = e - 1.$$

Thus,

$$\int_0^1 e^x \, dx \approx \frac{1 + 4e^{1/2} + e}{6}.$$

10 Summary

Simpson's rule is based on replacing f by a quadratic interpolating polynomial. Its basic form is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

It is exact for polynomials up to degree 3, and its error depends on the fourth derivative of the integrand.