Trapezoidal Rule with Step Size Control

1 Introduction

Numerical integration aims to approximate definite integrals of the form

$$I = \int_a^b f(x)\,dx,$$

especially when no closed-form expression is available.

One of the simplest methods is the **trapezoidal rule**, which approximates the integrand by linear interpolation.

2 Basic Trapezoidal Rule

For a single interval $[a, b]$, the trapezoidal rule is:

$$T = \frac{b-a}{2}\big(f(a) + f(b)\big).$$

3 Composite Trapezoidal Rule

Divide $[a, b]$ into n subintervals of equal width:

$$h = \frac{b-a}{n}, \quad x_i = a + ih.$$

Then:

$$T_n = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right].$$

4 Error Estimate

If $f \in C^2([a, b])$, the error satisfies:

$$|I - T_n| \le \frac{(b-a)}{12} h^2 \max_{x\in[a,b]} |f''(x)|.$$

Thus, the method is **second-order accurate**:

$$|I - T_n| = \mathcal{O}(h^2).$$

5 Step Size Control (Adaptive Refinement)

To achieve a desired accuracy, we refine the step size.

Let:

- T_h: trapezoidal approximation with step size h,
- $T_{h/2}$: approximation with half the step size.

Using error behavior:

$$I - T_h \approx Ch^2, \quad I - T_{h/2} \approx C\left(\frac{h}{2}\right)^2 = \frac{Ch^2}{4}.$$

Subtracting:

$$T_{h/2} - T_h \approx \frac{3}{4}Ch^2.$$

Hence, an error estimate is:

$$|I - T_{h/2}| \approx \frac{1}{3}|T_{h/2} - T_h|.$$

6 Adaptive Algorithm

Given tolerance ε:

1. Compute T_h with step size h.
2. Compute $T_{h/2}$ by halving the step size.
3. Estimate error:
$$E \approx \frac{1}{3}|T_{h/2} - T_h|.$$
4. If $E < \varepsilon$, accept $T_{h/2}$.
5. Otherwise, repeat with smaller h.

7 Example

Approximate:

$$I = \int_0^1 e^x \, dx = e - 1.$$

Step 1: $n = 1$

$$T_1 = \frac{1}{2}(f(0) + f(1)) = \frac{1}{2}(1 + e).$$

Step 2: $n = 2$

$$T_2 = \frac{1}{4}\big(f(0) + 2f(1/2) + f(1)\big).$$

Error Estimate

$$E \approx \frac{1}{3}|T_2 - T_1|.$$

Refine until the desired tolerance is achieved.