4. Differential equations and dynamic models

A differential equation (a DE) is an equation relating the independent variable (or variables), the dependent variable and its derivatives.

**Ordinary differential equation, ODE** is an equation for an unknown function \( x(t) \) of one independent variable \( t \):

\[
F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0.
\]

Example: \( \dot{x} = 3t^2 \).

**Partial differential equation, PDE** is an equation for an unknown function \( u \) of several independent variables

\[
F(x, y, u_x, u_y, u_{xx}, \ldots) = 0.
\]

We will not consider PDE’s, from now on DE means an ODE.

The order of a differential equation is the order of the highest derivative.
Differential equations are used for modelling a **deterministic process**: a law relating a certain quantity depending on some independent variable (for example time) with its rate of change, and higher derivatives.

**Examples**

**Newton's law of cooling:** $\dot{T} = k(T - T_\infty)$

where $T(t)$ is the temperature of a homogeneous body (can of beer) at time $t$, $T_0$ is the initial temperature at time $t_0 = 0$, $T_\infty$ is the temperature of the environment, $k$ is a constant (heat transfer coefficient)

**Radioactive decay:** $\dot{y}(t) = -ky(t), \quad k = \frac{\log 2}{t_{1/2}}$

where $y(t)$ is the remaining quantity of a radioactive isotope at time $t$, $t_{1/2}$ is the **half-life** and $k$ is the **decay constant**.

(Willard Libby, 1949, Nobel prize for chemistry in 1960 for **carbon dating**: dating objects containing organic material based on the ratio between the unstable isotope $^{14}\text{C}$ and the stable isotope $^{12}\text{C}$).

**Simple harmonic oscillator:** $\ddot{x} + \omega x = 0$

The function $x(t)$ is a **solution** of

$$F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0$$

on an interval $I$ if it is at least $n$ times differentiable and satisfies the identity

$$F(t, x(t), \dot{x}(t), \ddot{x}(t), \ldots, x^{(n)}(t)) = 0$$

for all $t \in I$.

Analytically solving a differential equation is typically very difficult, very often impossible.

To find approximate solutions we use simplifications, numerical methods, . . .
4.1. First order ODE’s

We will (mostly) consider first order ODE’s in the form

\[ \dot{x} = f(t, x) \]

- The **general solution** is a one-parametric family of solutions \( x = x(t, C) \).
- A **particular solutions** is a specific function from the general solution, that usually satisfies some *initial condition* \( x(t_0) = x_0 \).
- A **singular solution** is an exceptional solution that is not part of the general solution.

We will first look at some simple types of 1st order DE’s that are analytically solvable.

### 4.1.1 A separable DE is of the form

\[ \dot{x} = f(t)g(x) \]

This can be solved by inserting \( \dot{x} = \frac{dx}{dt} \), *separating variables* and integrating:

\[ \frac{dx}{g(x)} = f(t) \, dt \]

\[ \int \frac{1}{g(x)} \, dx = \int f(t) \, dt + C \]

Be careful: when dividing by \( g(x) \), check if \( g(x) = 0 \) is a solution of the equation.
Examples:

1. \( \dot{x} = kx \)

\[
\int \frac{dx}{x} = \int k\,dt + C, \text{ so } \log |x| = kt + C \text{ and } |x| = e^{kt+C}
\]

Clearly, \( x(t) = 0 \) is also a solution of the equation. By introducing a new constant \( e^C \) which, by abuse of notation, we again denote by \( C \), this is equivalent to

\[ x(t) = Ce^{kt}, C \in \mathbb{R}. \]

(This last argument will be implicitly used again in the next and several other examples.)

2. \( \dot{x} = kx(1-x) \)

\[
\int \frac{dx}{x(1-x)} = \int k\,dt + C, \text{ so } \log \left| \frac{x}{1-x} \right| = kt + C
\]

\[
\frac{x}{1-x} = Ce^{kt} \text{ and } x(t) = \frac{1}{Ce^{-kt} + 1}, C \in \mathbb{R}.
\]

\( x(t) \) is the \textit{logistic function}.

Two applications: \textit{1. population growth}

Let \( x(t) \) be the size of a population (bacteria, trees, hares, . . . ) at time \( t \).

The most common models for population growth are:

- \textit{exponential growth}: the growth rate is proportional to the size \( \dot{x} = kx \),
  
  the solution is the exponential function \( x(t) = x_0 e^{kt} \), where \( x_0 = x(0) \) is the initial population size.

- \textit{logistic growth}: the growth rate is proportional to the size and the resources \( \dot{x} = kx(1 - x/x_{\max}) \),
  
  where \( x_{\max} \) is the capacity of the environment, i.e. maximal population size that it still supports,
  
  the solution is the logistic function.

- \textit{general model}: the growth rate is proportional to the size, but the proportionality factor depends on time and size \( \dot{y} = k(x, t)f(x) \); the equation is not separable and is analytically solvable only in very specific cases.
2. Information spreading

\(x(t)\) is the part of a closed group that at time \(t\) knows a certain piece of information.

Let \(x_0 = x(t_0)\) be the informed part at time \(t = t_0\).

Consider two possible models:

- spreading through an external source: the rate of change is proportional to the uninformed part \(\dot{x} = k(1-x)\) with \(x_0 = 0\),
- spreading through "word of mouth" the rate of change is proportional to the number of encounters between informed and uninformed members \(\dot{x} = kx(1-x)\) logistic law, again, with \(x_0 > 0\).

4.1.2 A first order linear DE is of the form

\[ \dot{x} + f(t)x = g(t) \]

The equation is homogeneous if \(g(t) = 0\) and nonhomogenous if \(g(t) \neq 0\).

A homogeneous equation is separable, its general solution is always of the form \(C \chi_h(t)\), where \(C \in \mathbb{R}\) is a constant and \(\chi_h(t)\) is a particular solution.

The general solution of a nonhomogenous equation is a sum

\[ x(t) = x_p(t) + C \chi_h(t), \]

where \(C \chi_h\) is the general solution to the homogeneous equation, and \(x_p\) is any particular solution of the nonhomogenous equation.

The particular solution \(x_p\) can be obtained by variation of the constant, that is, by substituting the constant \(C\) is the homogenous solution by an unknown function \(C(t)\) which is then determined from the equation.
Example: \( t^2 \ddot{x} + tx = 1, \ x(1) = 2 \)

1. The general solution to the homogenous part

\[ t^2 \ddot{x} + tx = 0 \text{ is } x_h = \frac{C}{t}. \]

2. A particular solution of the nonhomogenous equation is obtained by variation of the constant:

\[ x = \frac{C(t)}{t}, \quad \dot{x} = \frac{C'(t)t - C(t)}{t^2} \]

by inserting into the equation we obtain

\[ C'(t)t - C(t) + C(t) = 1, \quad C'(t) = \frac{1}{t}, \quad C(t) = \log |t| \]

so the general solution of the nonhomogenous equation is

\[ x(t) = \frac{C}{t} + \frac{\log |t|}{t}. \]

3. Finally, since \( x(1) = C = 2 \), the solution is \( x(t) = \frac{2 + \log |t|}{t}. \)

A ball of mass \( m \) kg is thrown vertically into the air with initial velocity \( v_0 = 10 \) m/s. We follow its trajectory. By Newton’s second law of motion, \( ma = F \), where \( m \) is the mass, \( a = \dot{v} = \ddot{x} \) is acceleration and \( v \) velocity, and \( F \) is the sum of forces acting on the ball.

- Assuming no air friction the model is \( m\dot{v} = -mg \), where \( g \) is the gravitational constant.
- Assuming the linear law of resistance (drag) \( F_u = -kv \) the model is \( m\dot{v} = -mg - kv \).

Question: What will take longer: going up or falling down?

The second DE is nonhomogenous linear, so \( v = v_h + v_p \) where \( v_h = Ce^{-kt/m} \) and clearly \( v_p = -mg/k \) (the terminal speed) is a particular solution (no need for variation of constant...).
Motion of ball in the case $m = 1$, $k = 1$ and approximating $g \approx 10$ (we will omit units)

<table>
<thead>
<tr>
<th>Model</th>
<th>Velocity and position</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ma = -mg$</td>
<td>$\dot{v} = -10$</td>
<td>$v(t) = -10t + 10$</td>
</tr>
<tr>
<td>$ma = -mg - kv$</td>
<td>$\dot{v} = -v - 10$</td>
<td>$v(t) = 20e^{-t} - 10$</td>
</tr>
</tbody>
</table>

Answers: the ball reaches the top at time $t$ where $v(t) = 0$ and the ground at time $t$ where $x(t) = 0$.

- Assuming no friction, the ball is at the top at $t = 10$.
  
  At time $t = 20$, $x(t) = 0$, so it takes the same time going up and falling down.

- Assuming linear friction, the ball reaches the top at $t = \log 2$.
  
  At time $2 \log 2$, $x(2 \log 2) = 20 - 10 - 10 \log 2 > 0$ so it takes longer falling down than going up.
4.1.3 A **homogeneous** (nonlinear) DE is of the form

\[ \dot{x} = f \left( \frac{x}{t} \right). \]

The solution is obtained by introducing a new dependent variable \( u = x/t \). Then

\[ x = ut, \text{ and } \dot{x} = \dot{u}t + u, \]

and the DE for \( u \)

\[ \dot{u}t + u = f(u), \text{ so } t\dot{u} = f(u) - u \]

is a separable DE.

**Orthogonal trajectories**

Given a 1-parametric family of curves \( F(x, y, a) = 0 \), an **orthogonal trajectory** is a curve \( G(x, y) = 0 \) that intersects each curve from the given family at a right angle.

Assume that \( F \) is a differentiable function.

1. The family \( F(x, y, a) = 0 \) is the general solution of a 1st order DE, that is obtained by differentiating the equation (using implicit differentiation) and eliminating the parameter \( a \).

2. By substituting \( y' \) for \(-1/y'\) in the DE for the original family, we obtain a DE for curves with orthogonal tangents at every point of intersection.

3. The general solution to this equation is the family of orthogonal trajectories to the original equation.

Both families of curves together form an orthogonal net of coordinate curves in the plane.
Let us find the orthogonal trajectories to the family of circles through the origin with centers on the $y$ axis:

$$x^2 + y^2 - 2ay = 0.$$  

Differentiating this equation gives

$$2x + 2yy' - 2ay' = 0,$$  

so

$$a = \frac{x}{y'} + y.$$  

Inserting $a$ into the original equation we obtain the DE for the given family

$$x^2 - y^2 - 2xy \frac{y'}{y} = 0$$  

so

$$y' = \frac{2xy}{x^2 - y^2}.$$  

The DE for orthogonal trajectories is obtained by substituting $y'$ for $-1/y'$:

$$-\frac{1}{y'} = \frac{2xy}{x^2 - y^2},$$  

so

$$y' = -\frac{x^2 - y^2}{2xy}.$$  

This is a homogeneous DE:

$$y' = -\frac{x^2 - y^2}{2xy} = -\frac{x}{2y} + \frac{y}{2x}.$$  

By introducing $y = ux$ we obtain

$$u'x + u = -\frac{1}{2u} + \frac{u}{2},$$  

$$u'x = -\frac{1 + u^2}{2u}.$$  

$$\frac{2udu}{1 + u^2} = -\frac{dx}{x}$$  

$$\log (1 + u^2) = -\log x + \log C, \quad 1 + u^2 = \frac{C}{x},$$  

Inserting back $u = y/x$ gives the general solution

$$x^2 + y^2 = Cx.$$
Orthogonal trajectories to circles through the origin with centers on the \( y \) axis are circles through the origin with centers on the \( x \) axis.

Both families together form an orthogonal net:

4.1.4 We next consider \textit{exact} DE’s.

Notice first that a 1st order DE \( \dot{x} = f(t, x) \) can be rewritten in the form

\[
M(t, x) \, dt + N(t, x) \, dx = 0.
\]

The equation is \textit{exact} if there exists a differentiable function \( u(t, x) \) such that

\[
du = M(t, x) \, dt + N(t, x) \, dx,
\]

that is

\[
\frac{\partial u}{\partial t} = M(t, x) \text{ and } \frac{\partial u}{\partial x} = N(t, x).
\]

In this case, solutions of the equation are level curves of the function \( u \):

\[
u(t, x) = C.
\]
Recall from Calculus that if \( u \) has continuous second order partial derivatives then
\[
\frac{\partial u}{\partial x \partial t} = \frac{\partial u}{\partial t \partial x}.
\]
This gives the following necessary condition for exact differential equations
\[
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}.
\]
If \( M \) and \( N \) are differentiable for every \( (t, x) \in \mathbb{R}^2 \) this condition is also sufficient.

A **potential function** \( u \) can be computed from
\[
\int M(t, x) \, dt + C(x) = \int N(t, x) \, dx + C(t),
\]
where the integrating constant on the left can depend on \( x \) and on the right on \( t \).

Example: The DE \( x + ye^{2xy} + xe^{2xy}y' = 0 \) can be rewritten as
\[
(x + ye^{2xy}) \, dx + xe^{2xy} \, dy = 0.
\]
The equation is exact since
\[
\frac{\partial (x + ye^{2xy})}{\partial y} = \frac{\partial (xe^{2xy})}{\partial x} = (e^{2xy} + 2xye^{2xy}).
\]
A potential function is equal to
\[
u(x, y) = \int (x + ye^{2xy}) \, dx = \frac{x^2}{2} + \frac{1}{2} e^{2xy} + D(y)
\]
\[
= \int (xe^{2xy}) \, dy = \frac{1}{2} e^{2xy} + C(x),
\]
It follows that \( C(x) = x^2/2, \) and
\[
u(x, y) = \frac{x^2}{2} + \frac{1}{2} e^{2xy}.
\]
The general solution is the family of level curves
\[
\frac{x^2}{2} + \frac{1}{2} e^{2xy} = E, \quad E \in \mathbb{R}.
\]
4.1.5 Geometric picture:

Let \( D \subset \mathbb{R}^2 \) be the domain of the function \( f(x, y) \). For each point \((x, y) \in D\) the DE
\[
y' = f(x, y)
\]
gives the value \( y' \) of the coefficient of the tangent to the solution \( y(x) \) through this specific point, that is, the direction in which the solution passes through the point.

All these directions together form the *directional field* of the equation.

A solution of the equation is represented by a curve \( y = y(x) \) that follows the given directions at every point \( x \), that is, the coefficient of the tangent corresponds to the value \( f(x, y(x)) \).

The general solution to the equation is a family of curves that each follows the given directions.

Directional fields and solutions of
\[
y' = ky \quad \quad y' = ky(1 - y)
\]
Theorem (Existence and uniqueness of solutions)

If \( f(x, y) \) is continuous and differentiable with respect to \( y \) on the rectangle

\[
D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b], \quad a, b > 0
\]

then the DE with initial condition

\[
y' = f(x, y), \quad y(x_0) = y_0,
\]

has a unique solution \( y(x) \) defined at least on the interval

\[
[x_0 - \alpha, x_0 + \alpha], \quad \alpha = \min \left\{ a, \frac{b}{M}, \frac{1}{N} \right\},
\]

where

\[
M = \max \{ f(x, y), (x, y) \in D \} \quad \text{and} \quad N = \max \left\{ \frac{\partial f(x, y)}{\partial y}, (x, y) \in D \right\}.
\]

4.1.6 Numerical methods

Given the DE with initial condition

\[
y'(x) = f(y, x), \quad y(x_0) = y_0.
\]

Instead of analytically finding the solution \( y(x) \), we construct a recursive sequence of points

\[
x_i = x_0 + ih, \quad y_i \approx y(x_i), \quad i \geq 0
\]

where \( y_i \) is an approximation to the value of the exact solution \( y(x_i) \), and \( h \) is the step size.

A number of numerical methods exists, the choice depends on the type of equation, desired accuracy, computational time,...

We will look at the simplest and best known.
Euler’s method is the simplest and most intuitive approach to numerically solve a DE.

At each step the value \( y_{i+1} \) is obtained as the point on the tangent to the solution through \((x_i, y_i)\) at \( x_{i+1} = x_i + h \):

\[
\begin{align*}
\text{initial condition: } & (x_0, y_0) \\
\text{for each } & i: x_{i+1} = x_i + h, \\
& y_{i+1} = y_i + hf(x_i, y_i).
\end{align*}
\]

The point \((x_{i+1}, y_{i+1})\) typically lies on a different particular solution than \((x_i, y_i)\), at each step, the error is of order \( O(h^2) \). The cumulative error grows with each step.

The Runge-Kutta method is probably the most widely used numerical method for DE’s:

\[
\begin{align*}
\text{initial condition: } & (x_0, y_0) \\
\text{at each step } & x_i \text{ is computed as a weighted average of approximations at } x = x_i, x = x_i + h/2 \text{ and } x = x_{i+1}: \\
& x_{i+1} = x_i + h, \quad y_{i+1} = y_i + (k_1 + 2k_2 + 2k_3 + k_4)/6, \\
& k_1 = hf(x_i, y_i), \\
& k_2 = hf(x_i + h/2, y_i + k_1/2), \\
& k_3 = hf(x_i + h/2, y_i + k_2/2) \text{ in} \\
& k_4 = hf(x_i + h, y_i + k_3)
\end{align*}
\]

The error at each step is of order \( O(h^5) \). The cumulative error is of order \( O(h^4) \).
Below is a comparison of Euler’s and Runge-Kutta methods for the DE

\[ y' = -y - 1 \quad y(0) = 1 \] with step size \( h = 0.3 \).

The red curve is the exact solution \( y = 2e^{-x} - 1 \).