GRAPH AND SEMIGROUP HOMOMORPHISMS ON NETWORKS OF RELATIONS

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The algebraic definitions presented here are motivated by our search for an adequate formalization of the concepts of social roles as regularities in social network patterns. The theorems represent significant homomorphic reductions of social networks which are possible using these definitions to capture the role structure of a network. The concepts build directly on the pioneering work of S.F. Nadel (1957) and the pathbreaking approach to blockmodeling introduced by Lorrain and White (1971) and refined in subsequent years (White, Boorman and Breiger 1976; Boorman and White 1976; Arabie, Boorman and Levitt, 1978; Sailer, 1978).

Blockmodeling is one of the predominant techniques for deriving structural models of social networks. When a network is represented by a directed multigraph, a blockmodel of the multigraph can be characterized as mapping points and edges onto their images in a reduced multigraph. The relations in a network or multigraph can also be composed to form a semigroup.

In the first part of the paper we examine "graph" homomorphisms, or homomorphic mappings of the points or actors in a network. A family of basic concepts of role equivalence are introduced, and theorems presented to show the structure preserving properties of their various induced homomorphisms. This extends the "classic" approach to blockmodeling via the equivalence of positions.

Lorrain and White (1971), Pattison (1980), Boyd (1980, 1982), and most recently Bonacich (1982) have explored the topic taken up in the second part of this paper, namely the homomorphic reduction of the semigroup of relations on a network, and the relation between semigroup and graph homomorphisms. Our approach allows us a significant beginning in reducing the complexity of a multigraph by collapsing relations which play a similar "role" in the network.

1. Networks with single relations

1.1. Graphs and their images

Definition 1. A *graph* (usually referred to as a digraph) is an ordered pair

$$G = \langle P, R \rangle$$

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where P is a finite set of points (points, objects, actors), and R is a relation (type of tie) on P, that is, a subset of the ordered pairs of points in $P \times P$.

Definition 2. A *function* $f: P \to P'$ is a mapping of each element a in the set P to an image element $f(a)$ in the set P'. An "onto" function (surjection) is a mapping where all elements in P' are images of elements in P.

Definition 3. An *equivalence* $\equiv$ on P is a relation such that for all a, b, c in P,

$a \equiv a$

$a \equiv b$ implies $b \equiv a$

$(a \equiv b$ and $b \equiv c)$ implies $a \equiv c$.

These are the properties of reflexivity, symmetry, and transitivity.

Lemma 1. Every function $f: P \to P'$ induces an equivalence relation $\equiv_f$ on P, namely, for each a, $b \in P$,

$a \equiv_f b$ if and only if $f(a) = f(b)$.

The proof of this lemma is found in standard texts. All other proofs of lemmas and theorems will be given in the Appendix.

Definition 4. Let $G = \langle P, R \rangle$ be a graph and $f: P \to P'$ be an onto map. Let R' be the relation on P' defined by $R' = \{\langle f(a), f(b) \rangle : \langle a, b \rangle \in R\}$. Then R' is called the relation on P' *induced by R and f*.

1.2. Graph homomorphisms and blockmodel images

Homomorphisms are mappings which preserve structure. The minimal graph homomorphism is a function which maps the points of a graph into points in an image of the graph, and preserves edges or connections as image edges or connections. This may be expressed diagrammatically:

Graph: Image:
Every edge $----f---\to$ Edge   (Homomorphism)

Different types of homomorphisms preserve additional features of the structure of a graph. The structure which is preserved may be defined by the properties of f in terms of its inverse mapping f^{-1} from the image to the preimage for different types of homomorphisms.

Formal definitions for the types of homomorphisms which are most useful in the network analysis of role structures are given below. We begin with the full homomorphism in which every edge in the image is induced by some edge in the preimage. Since there are no extraneous edges in the image, this and each of the homomorphisms which follow generates a structural model of a network.

Regular and structural homomorphisms are models of particular importance for the study of role systems. In the case of a regular homomorphism, points having the same image necessarily occupy the same abstract position or "role" in the total network or graph. Two points have the same (role) image in a regular homomorphism if and only if, given that one has a relation with a point in a second image set (or role), the other has an identical relation with a counterpart in that set. This is the principle of role parallels. Two points have the same image in a structural homomorphism if and only if they are identically related to all other points.

Definition 5. Let $G = \langle P, R \rangle$ and $G' = \langle P', R' \rangle$ be two graphs. Then $f: G \to G'$ is a *full graph homomorphism* if and only if $f: P \to P'$ is an onto map such that for all $a, b \in P$ and $x, y \in P'$

aRb implies $f(a) R' f(b)$,

and

$xR'y$ implies there exist $c, d \in P$ such that $cRd, f(c) = x$, and $f(d) = y$.

The full homomorphic image of a graph is termed a *blockmodel* by Arabie, Boorman and Levitt (1978: 31–32).

Proposition A. Let $G = \langle P, R \rangle$ be a graph and $f: P \to P'$ be an onto map for some set P'. If R' is the relation on P' induced by f and R, and $G' = \langle P', R' \rangle$ then the map $f: G \to G'$ is a *full graph homomorphism*.

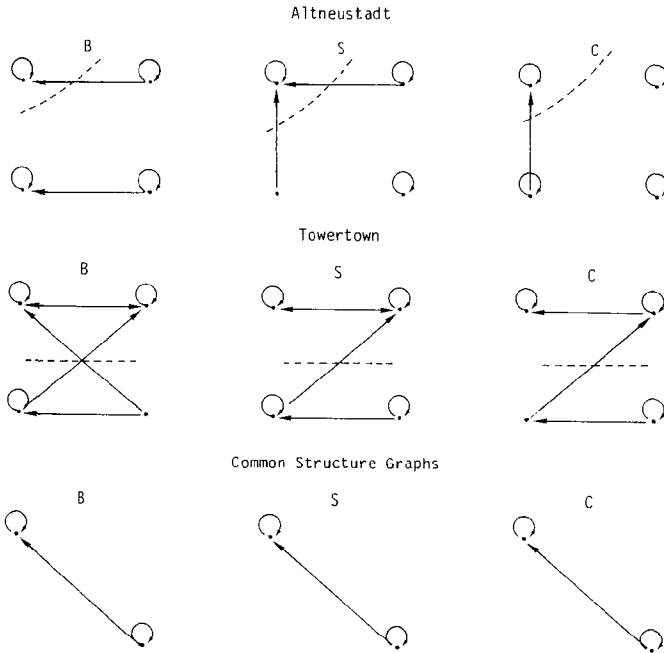Full homomorphisms are useful in structural comparisons of net-

Figure 1. Illustrative graphs from Breiger and Pattison (1978) *Source*: Bonacich (1981).

works or graphs. An example of the steps in this process derives from work by Laumann and Pappi (1976) on relations between elite members of two cities, 'Altneustadt' in Germany and 'Towertown' (pseudonyms) in the U.S. Breiger and Pattison (1978) blockmodeled structurally equivalent actors (see Definitions 10 and 25) to derive summary graphs of the role structures of the communities in terms of three relations: business (B), community affairs discussions (C), and social contacts (S), as shown in the graphs in Figure 1. Bonacich (1981) then compared these two sets of graphs by using a full homomorphism from the graphs for each city to 'common structure' graphs, also shown in Figure 1. These graphs show shared aspects of the leadership structure in the two communities. The example shows how stronger homomorphisms may be employed at one stage in the analysis to reveal particular features of social networks, while weaker homomorphisms are employed at a later stage to show more generic features. The advantages of a family of homomorphic modeling tools, from stronger to

weaker, ought to be obvious in terms of different levels of generality.

The full homomorphism is useful for analysis of common structure (Bonacich 1981) but is too general to identify more precise role positions. For example, any two non-empty graphs have the same full homomorphic image of a single point image-connected with itself. Unconnected as well as connected points in each graph are mapped onto this same point in the image. Clearly, this does not correspond to mapping points to common roles or positions in a network.
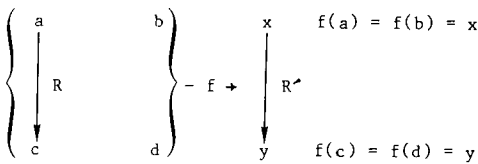
The regular homomorphism, however, can be thought of as a mapping of points in a graph onto distinct roles or positions, with the proviso that if two roles are image-connected, then an incumbent of one of the roles will be connected to some alter who is mapped onto the other role. This is formalized as follows.

Definition 6. A full graph homomorphism $f: G \rightarrow G'$ is *regular* if and only if for all $a, b \in P$,

$f(a) R' f(b)$ implies there exist $c, d \in P$ such that

$cRb, aRd, f(c) = f(a)$, and $f(d) = f(b)$.

Not every full graph homomorphism is a regular graph homomorphism, as is shown by the following example. Here $\{a, b, c, d\}$ are points in the original graph, and the function f maps these points into the set $\{x, y\}$ carrying over any of the interpoint connections onto the image graph.

$$\left\{ \begin{matrix} a & & b \\ \Big| & & \\ \Big| R & & \\ \Big\downarrow & & \\ c & & d \end{matrix} \right\} - f \rightarrow \begin{matrix} x & f(a) = f(b) = x \\ \Big| & \\ \Big| R' & \\ \Big\downarrow & \\ y & f(c) = f(d) = y \end{matrix}$$

Example 1

Note that $f(b) R' f(d)$ but bRx is true for no x.

Regular homomorphisms require that occupants of one role will be identically connected to some occupants of a "counterpart" role. In role systems, it is not expected that all occupants of one role will be identically connected to occupants of a "counterpart" role. This more stringent requirement is the basis for the analysis of role systems by

Harrison White and associates. There are some circumstances or types of roles in which it is expected that all occupants of one role will be identically connected to occupants of counterparts roles. This may be formally defined in terms of a structural homomorphism familiar to graph theory (Hedetniemi, 1966; Lorrain 1974; Arabie, Boorman and Levitt 1978), as follows.

Definition 7. A full graph homomorphism $f: G \rightarrow G'$ is *structural* if and only if for all a, b in P where $a \neq b$,

$f(a)R'f(b)$ implies aRb.

Not every regular graph homomorphism is a structural homomorphism, as is shown by Example 2.
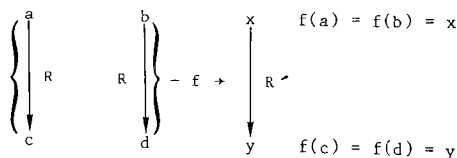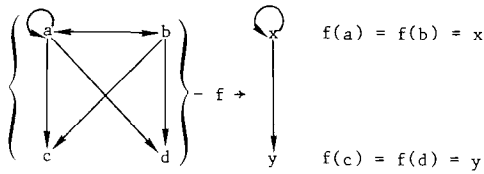

Example 2

Note that $f(a)R'f(d)$ but aRd is false.

In structural homomorphisms the fact that the image of a point is image-connected to itself does not imply that its preimage is connected to itself. The strongest of the graph homomorphisms generalizes the concept of structural equivalence to include reflexivity. If the connection of a point with itself is significant, then the image-connection of a point implies that it is connected to itself in the preimage. This is formalized as follows.

Definition 8. A full graph homomorphism $f: G \rightarrow G'$ is *strong* if and only if for all a, b in P,

$f(a)R'f(b)$ implies aRb.

Not every structural graph homomorphism is strong, as illustrated by Example 3.

Example 3

Note that $f(b)R'f(b)$ but bRb is false.

Strong homomorphisms are the basis for certain multidimensional spatial models of graphs or networks. Guttman (1977) shows that symmetric graphs can be represented in a multidimensional space where two points are mapped to the same image in the space if and only if they have identical connections to other points, to each other, and with themselves (the distance from a point in this space to itself is zero). He also notes how such spatial representations can be generalized to asymmetric graphs. These ideas are explored in more detail by Freeman (1983).

The four graph homomorphisms in Definitions 5–8 are of ascending strength in the sense that the stronger imply the weaker, as stated in the following theorem.

Theorem 1. If $f: G \rightarrow G'$ then
 (i) f is a strong graph homomorphism implies f is a structural homo-
    morphism;
 (ii) f is a structural homomorphism implies f is a regular homomor-
    phism.

Full and strong homomorphisms are defined in Grätzer (1979: 81) for partial algebras. The regular and structural homomorphisms as defined here are used for analyzing specific aspects of role structure.

1.3. Equivalences

Recall that every graph homomorphism induces an equivalence on the domain set (Lemma 1). The following theorems show that each type of graph homomorphism induces a particular kind of equivalence and conversely that each special type of equivalence is induced by a graph homomorphism of its associated type.

Definition 9. If $G = \langle P, R \rangle$ and $\equiv$ is an equivalence relation on P, then $\equiv$ is a *strong equivalence* if and only if for all $a, b, c \in P$, $a \equiv b$ implies
  (i) aRb if and only if bRa;
 (ii) aRc if and only if bRc; and
(iii) cRa if and only if cRb.

Strongly equivalent points are related in the same way to themselves, to each other, and to every point.

Definition 10. If $G = \langle P, R \rangle$ and $\equiv$ is an equivalence relation on P then $\equiv$ is a *structural equivalence* if and only if for all $a, b, c \in P$ such that $a \neq c \neq b$, $a \equiv b$ implies
   (i) aRb if and only if bRa;
  (ii) aRc if and only if bRc;
 (iii) cRa if and only if cRb; and
 (iv) aRa implies aRb.

Structurally equivalent points are related in the same way to each other and to all other points.

Definition 11. If $G = \langle P, R \rangle$ and $\equiv$ is an equivalence relation on P then $\equiv$ is a *regular equivalence* if and only if for all $a, b, c \in P$, $a \equiv b$ implies
 (i) aRc implies there exists $d \in P$ such that bRd and $d \equiv c$; and
(ii) cRa implies there exists $d \in P$ such that dRb and $d \equiv c$.

Regularly equivalent points are connected in the same way to matching equivalents.

Theorem 2A. The equivalence induced by a strong graph homomorphism is a strong equivalence and conversely every strong equivalence is induced by some strong graph homomorphism.

Theorem 2B. The equivalence induced by a structural graph homomorphism is a structural equivalence relation and conversely every structural equivalence relation is induced by some structural homomorphism.

Theorem 2C. The equivalence induced by a regular graph homomorphism is a regular equivalence relation and conversely every regular equivalence relation is induced by some regular homomorphism.

Equivalence relations on a set P can be thought of as subsets of $P \times P$. As such, they are partially ordered by set inclusion. A collection of equivalences has a maximal element if there is one equivalence relation in the collection which contains all the rest.

Theorem 3A. The collection of all strong equivalence relations on a graph has a maximal element.

Theorem 3B. The collection of all structural equivalence relations on a graph has a maximal element.

Theorem 3C. The collection of all regular equivalence relations on a graph has a maximal element.

1.4. The semigroup of compound relations

Let R be a relation and define $\circ$ as composition of relations, *i.e.*, if aRb and bRc then $a(R \circ R)c$, so

$$R \circ R = \{\langle a, c \rangle : \text{there exists } b \in P \text{ such that } \langle a, b \rangle \text{ and } \langle b, c \rangle \in R\}.$$

The operation "$\circ$" is associative on the set S of all relations on P generated by R. In other words, $\langle S, \circ \rangle$ is a semigroup. Note that elements R^n and R^m in S are equal if they contain the same set of ordered pairs in $P \times P$.

Let $f: G \to G'$ be a full graph homomorphism from $\langle P, R \rangle$ to $\langle P', R' \rangle$. Now for each relation $Q \in S$ let Q' be the corresponding relation on P' induced by f and Q (see Definition 4). Then let

$$S' = \{Q' : Q \in S\},$$

and

$$\hat{f}: S \to S'$$

such that

$$\hat{f}(Q) = Q'.$$

Theorem 4. If $f: G \to G'$ is a regular, structural, or strong graph homomorphism (with respect to R) then f is regular, structural, or strong, respectively, for any relation in S. That is $f: \langle P, Q \rangle \to \langle P', Q' \rangle$ is regular, structural, or strong respectively for any $Q \in S$.

Theorem 5. If $f: G \to G'$ is a regular graph homomorphism, then $\hat{f}: S \to S'$ is a semigroup homomorphism. That is

$$\hat{f}(Q_1 \circ Q_2) = \hat{f}(Q_1) \circ \hat{f}(Q_2) = Q_1' \circ Q_2'.$$

*Theorem 6.* If $f: G \to G'$ is a strong graph homomorphism, then $\hat{f}: S \to S'$ is a semigroup isomorphism.

The above theorem does not hold if f is only a structural homomorphism. This can be seen in Example 3 (above). Note that for points $r \neq s$, $f(r)R'f(s)$ implies rRs. However, $f(b)R'f(b)$ but not bRb, so f is structural but not strong. Note also that $\langle b, b \rangle \notin R$ but $\langle b, b \rangle \in R^2$, so $R \neq R^2$. However, $R' = (R')^2$ so $f: S \to S'$ is not an isomorphism.

Definition 12. A graph G is *acyclic* if and only if $\langle a, a \rangle \notin R^n$ for all $a \in P$ and $n \in Z^+$.

Theorem 7. If $f: G \to G'$ is a structural graph homomorphism and G is an acyclic graph, then $\hat{f}: S \to S'$ is a semigroup isomorphism.

Strong homomorphisms of a graph preserve the exact structure of the semigroup of relations generated by the relation on the graph. Structural homomorphisms preserve this structure only when all of the relations are irreflexive. Regular homomorphisms do not necessarily preserve exact semigroup structure, however, as is seen in the following example.
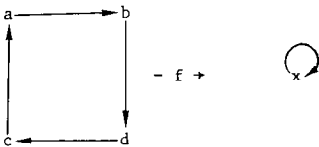
Example 4

Semigroup Composition Table:

$\circ$	R	RR	RRR	I
R	RR	RRR	I	R
RR	RRR	I	R	RR
RRR	I	R	RR	RRR
I	R	RR	RRR	I

$$-\hat{f} \rightarrow \begin{array}{c|c} \circ & I \\ \hline I & I \end{array}$$

2. Networks with attributes and multiple relations

To generalize fully the use of these definitions, we want to define networks as multigraphs (with multiple relations), where the nodes of the network also have attributes. We can then define a network algebra on which our homomorphisms and equivalences operate.

2.1. Node attributes: Equivalences and class identities

Definition 13. An *attribute equivalence* on a graph G is the equivalence relation $\equiv A$ induced by a subset A of P having a given attribute, where

$$\equiv A = \{\langle i, j \rangle : i, j \in A \text{ or } i, j \notin A\}.$$

Lemma 2. If $G = \langle P, \equiv A \rangle$ is the graph of an attribute equivalence relation $\equiv A$, $\equiv A$ is also the equivalence induced by the largest regular (structural or strong) homomorphism of the graph.

Definition 14. A *class identity* on a graph G is a subset I_A of the identify relation $I = \{\langle i, i \rangle : i \in P\}$ for a set A of nodes in P having a given class attribute, thus

$$I_A = \{\langle i, i \rangle : i \in A \subseteq P\}.$$

Lemma 3. If $G = \langle P, I_A \rangle$ is the graph of a class identity I_A defined by the class attribute A, then the attribute equivalence $\equiv A$ is also the equivalence induced by the largest regular homomorphism of the graph.

2.2. Networks and multiple relations

Definition 15.   A *network* is an ordered pair $N = \langle P, \mathfrak{R} \rangle$ where P is a set of points and $\mathfrak{R}$ is a collection of relations on P.

A network is sometimes called a (directed) multigraph and includes relations on P which may be attribute equivalences or class identities. As with graphs defined by a single relation, homomorphisms can be defined on networks. A network homomorphism involves two mappings. The number of variations possible for defining stronger mappings is greatly increased.

Definition 16a.   Let $N = \langle P, \mathfrak{R} \rangle$ and $N' = \langle P', \mathfrak{R}' \rangle$ be two networks. A *weak full network homomorphism* $f: N \to N'$ is an ordered pair of mappings $f = \langle f_1, f_2 \rangle$ such that $f_1: P \to P'$ and $f_2: \mathfrak{R} \to \mathfrak{R}'$ are onto, for every $a, b \in P$ and $R \in \mathfrak{R}$, aRb implies $f_1(a)f_2(R)f_1(b)$, and for every $x, y \in P'$ and $R' \in \mathfrak{R}'$, $xR'y$ implies there exists $c, d \in P$ and $R \in \mathfrak{R}$ such that $f_1(c) = x, f_1(d) = y, f_2(R) = R'$ and cRd.

Definition 16b.   Let $N = \langle P, \mathfrak{R} \rangle$ and $N' = \langle P', \mathfrak{R}' \rangle$ be two networks. A *full network homomorphism* $f: N \to N'$ is an ordered pair of mappings $f = \langle f_1, f_2 \rangle$ such that $f_1: P \to P'$ and $f_2: \mathfrak{R} \to \mathfrak{R}'$ are onto, for every $a, b \in P$ and $R \in \mathfrak{R}$, aRb implies $f_1(a)f_2(R)f_1(b)$, and for every $x, y \in P'$ and $R \in \mathfrak{R}$, $xf_2(R)y$ implies there exist $c, d \in P$ such that $f_1(c) = x, f_1(d) = y$ and cRd.

Note that for each relation $R \in \mathfrak{R}$, the full network homomorphism $f: N \to N'$ induces a full graph homomorphism from $\langle P, R \rangle$ to $\langle P', f_2(R) \rangle$. An example of two forms of regular homomorphisms follow.

Definition 17a.   A weak full network homomorphism $f: N \to N'$ is a *weak regular network homomorphism* if for each $R' \in \mathfrak{R}'$, $f_1(a)R'f_1(b)$ implies there exist $c, d \in P$ and $R, Q \in \mathfrak{R}$ such that $f_1(a) = f_1(c), f_1(b) = f_1(d), f_2(Q) = f_2(R), f_2(R) = R', cRb$, and aQd for all $a, b \in P$.

Definition 17b.   A full network homomorphism $f: N \to N'$ is a *regular network homomorphism* if for each $R \in \mathfrak{R}$, $f_1(a)f_2(R)f_1(b)$ implies there exist $c, d \in P$ such that $f_1(a) = f_1(c), f_1(b) = f_1(d), cRb$ and aRd for all $a, b \in P$.

These two pairs of definitions treat differently the case where f_2 is not one-to-one. If two or more relations have the same image, the first of each pair of definitions requires that the homomorphism be full or regular on their union. The second definition requires that the homomorphism be full or regular on each of the relations in $\mathcal{R}$. Similar variations can be made on the network homomorphisms corresponding to the structural, and strong graph homomorphisms. If we restrict our attention to only the strong version of each of these definitions, they can be defined simply by requiring that the induced graph homomorphisms meet the property of being structural or strong. The theorems for both homomorphisms and equivalences on graphs will also now generalize to corresponding theorems for networks including those which involve composition of relations and their semigroups.

2.3. Connectivity

Definition 18. A full network homomorphism is *connectivity preserving* if and only if for any sequence of points $x_1, \ldots, x_{n+1} \in P$ and relations $R_1, \ldots, R_n$ such that $f_1(x_1)f_2(R_1)f_1(x_2)\ldots f_1(x_n)f_2(R_n)f_1(x_{n+1})$ there exist points $y_2, \ldots, y_{n+1}$ such that $f_1(x_2) = f_1(y_2), \ldots, f_1(x_{n+1}) = f_1(y_{n+1})$ and $x_1 R_1 y_2 R_2 \ldots R_n y_{n+1}$ and there also exist points $z_1, \ldots, z_n$ such that

$$f_1(x_1) = f_1(z_1), \ldots, f_1(x_n) = f_1(z_n) \text{ and } z_1 R_1 z_2 R_2 \ldots z_n R_n x_{n+1}.$$

A sequence of related points is a path. The above definition states that a path in the image graph corresponds to specific paths in the preimage passing through each point mapped onto an endpoint of the image path.

Theorem 8. If f is a regular network homomorphism then f is connectivity preserving.

Definition 19. A full network homomorphism is *strongly connectivity preserving* if and only if for every sequence of points $x_1, \ldots, x_{n+1}$ (where $x_i \neq x_j$ for $i \neq j$) and relations $R_1, \ldots, R_n$, $f_1(x_1)f_2(R_1)f_1(x_2)\ldots f_1(x_n)f_2(R_n)f_1(x_{n+1})$ implies $x_1 R_1 x_2 R_2 \ldots R_n x_{n+1}$.

Theorem 9. If f is a structural network homomorphism then f is strongly connectivity preserving.

2.4. Multiplex graphs and bundle equivalence

In a network a particular ordered pair of points may be an element of more than one relation. We call the set of all relations which contain the pair of points $\langle a, b \rangle$ the *bundle* of relations for that pair. That is $B_{ab} = \{R \in \mathfrak{R} : aRb\}$. A second ordered pair of points $\langle c, d \rangle$ may share the same bundle, that is $B_{ab} = B_{cd}$. It may be that the pair $\langle a, b \rangle$ is not related by any members of $\mathfrak{R}$, so that $B_{ab} = \emptyset$. Let B^* be the collection of all non-empty bundles. We can now define new relations on P by looking at the pairs of points that share bundles. These multiplex relations are of substantive interest in social theory.

Definition 20. Let $B \in B^*$. Then the relation $M_B = \{\langle a, b \rangle : B_{ab} = B\}$ is called a *multiplex* relation *induced* by the network $N = \langle P, \mathfrak{R} \rangle$.

For each ordered pair $\langle a, b \rangle$ there is a unique bundle associated with it. This bundle may be either empty or a member of B^*. This implies that either $\langle a, b \rangle$ is a member of no M_B or has only one such multiplex relation. The collection $\mathfrak{M}$ of all such relations induced on a given network define a special type of graph.

Definition 21. A *multiplex graph* is a network $C = \langle P, \mathfrak{M} \rangle$ such that for each pair of relations $M_1, M_2 \in \mathfrak{M}$, $M_1 \cap M_2 = \emptyset$.

The graph consisting of the points from a given network and the multiplex relations induced on that network is a multiplex graph. The procedure for moving from a given network to its induced multiplex graph gives a unique result. However, as the following example shows, the properties of network homomorphisms do not carry over to the homomorphisms induced on the multiplex graph. In Example 5, $N = \langle P, \mathfrak{R} \rangle$ where $P = \{a, b, c, d\}$ and $\mathfrak{R} = \{R, S\}$ and $N' = \langle P', \mathfrak{R}' \rangle$ with $P' = \{x, y\}$ and $\mathfrak{R}' = \{R'\}$. Note that f is regular as a homomorphism from N to N'. The multiplex graph is $C = \langle P, \mathfrak{M} \rangle$ where $\mathfrak{M} = \{M_1, M_2, M_3\}$. The map $f_1 : P \to P'$ along with the requirement that f be at least a full network homomorphism gives an induced collection of relations on P' namely $\mathfrak{M}' = \{M_1'\}$. To emphasize that the map from C to $C' = \langle P', \mathfrak{M}' \rangle$ is a different network homomorphism, we have labeled it f_M. Even though f is regular, f_M is not. Regularity however is preserved in the opposite direction:
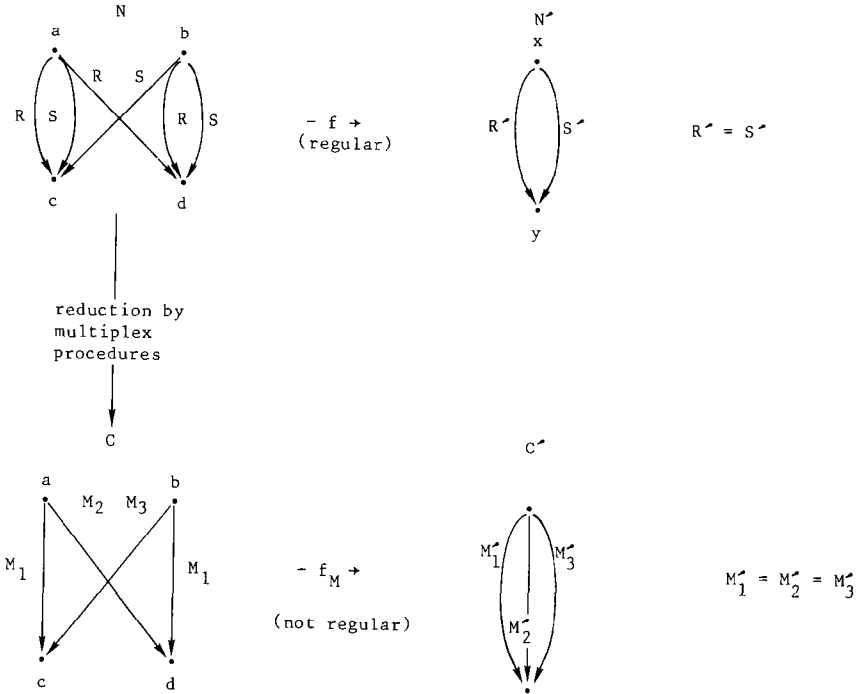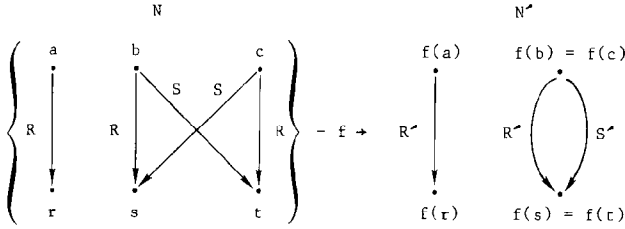
Example 5

Theorem 10. If $N = \langle P, \mathfrak{R} \rangle$ is a network, $C = \langle P, \mathfrak{M} \rangle$ the multiplex graph derived from it, and $f: C \to C' = \langle P', \mathfrak{M}' \rangle$ a full network homomorphism, then f induces a full network homomorphism on N and

(i) if f is regular the induced homomorphism is regular;

(ii) if f is strong the induced homomorphism is strong.

The image of a multiplex graph under a regular network homomorphism is not necessarily a multiplex graph. This is shown in Example 6, where a pair of image points are connected by more than one image relation, and the image relations R' and S' are distinct in their ordered pairs in $P' \times P'$.

Multiplex graphs give a representation of the unique bundles of relations and shared attributes between individuals which are used to define roles. Mandel and Winship (1979) suggest that occupancy of the
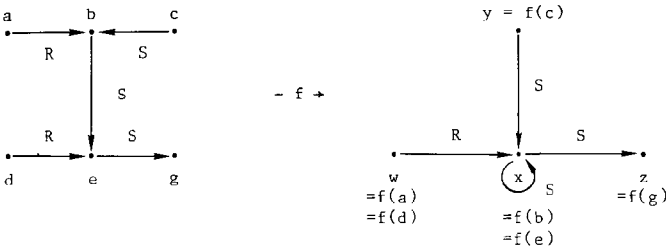
Example 6

same role is partly captured by the equivalence of points in a network in terms of these bundles. Using our definition of a multiplex graph, we restate their definition of local role equivalence as follows:

Definition 22. If $C = \langle P, \mathfrak{M} \rangle$ is a multiplex graph and $\equiv$ is an equivalence on P then $\equiv$ is a *bundle equivalence* if and only if for all $a, b, c \in P$ and $M \in \mathfrak{M}$, $a \equiv b$ implies
 (i) aMc if and only if there exists d such that bMd;
 (ii) cMa if and only if there exists d such that dMb

A *bundle homomorphism* is a full network homomorphism which identifies the points of the equivalence classes defined by bundle equivalences, and where the mapping of relations is the identity. This reduction of graphs under bundle equivalence is not necessarily a regular homomorphism, as shown below. Here f is a bundle homomor-

Example 7

phism but is not regular. Regular, structural and strong homomorphisms on a network are not necessarily bundle homomorphisms because the former allow collapsing of relations while the latter does not.

However, a strong homomorphism on a network or multiplex graph where relations are not collapsed is a bundle homomorphism.

While the local structure of social roles is captured in a bundle homomorphism by putting together points with the same patterns of incoming and outgoing arrows, the global or relational structure of their roles is not captured. In the image of the above graph, on Example 7, $f(b)$ is connected to $f(g)$ by S. In the preimage, however, we see that b is not connected by S to any point equivalent to g.

To capture the global role structure of multiplex social relationships, we require a stronger homomorphism, one which has properties of both the bundle homomorphism and the regular homomorphism. The strong network homomorphism has both properties, but is too restrictive. A weaker homomorphism with both properties is defined below.

Defined 23. Let $f: N \to N'$ be a regular network homomorphism. f is a *juncture network homomorphism* if and only if for all a, b, c, d, in P

$$f_1(a) = f_1(c) \text{ and } f_1(b) = f_1(d) \text{ implies } B_{ab} = B_{cd}, B_{ab} = \emptyset \text{ or } B_{cd} = \emptyset.$$

Note that this also implies that $B_{ab} = B_{cb} = B_{cd} = B_{ad}$, or some are empty. This restriction guarantees that for this specialization of a regular homomorphism, there is a unique multiplex relation, if any, between every role pair.

Theorem 11a.. Every strong network homomorphism is a juncture network homomorphism.

Theorem 11b. Let $f = \langle f_1, f_2 \rangle$ be a juncture homomorphism. Then $f^* = \langle f_1, f_3 \rangle$, where f_3 is the identity, is a bundle homomorphism.

Example 8 below gives two juncture homomorphisms which are not strong. Example 7 gives a bundle homomorphism which is not juncture although the mapping of relations is the identity. For a network in which the collection of relations consists of only one relation, every regular homomorphism is trivially a juncture homomorphism. On the other hand the homomorphism $f: N \to N'$ in Example 6 is regular but not juncture.

The next collection of theorems shows that juncture homomorphisms have the desired properties of preserving multiplexity and preserving

the semigroup of relations without the restrictiveness of a strong homomorphism.

Theorem 12. Let $f: N \to N'$ be a juncture network homomorphism and C the multiplex graph derived from N. Then f induces a map from C to C' and C' is a multiplex graph.

Thus the image of a multiplex graph under a juncture (unlike the regular) network homomorphism is a multiplex graph. Juncture homomorphisms share with strong homomorphisms the property of preserving composition of relations, as the following theorem shows.

Theorem 13. Let $f: N \to N'$ be a juncture network homomorphism where $N = \langle P, \mathcal{R} \rangle$, $N' = \langle P', \mathcal{R}' \rangle$ and $f = \langle f_1, f_2 \rangle$. If $\circ$ is relation composition and $\langle \mathcal{R}, \circ \rangle$ is a semigroup then $f_2: \langle \mathcal{R}, \circ \rangle \to \langle \mathcal{R}', \circ \rangle$ is an isomorphism.
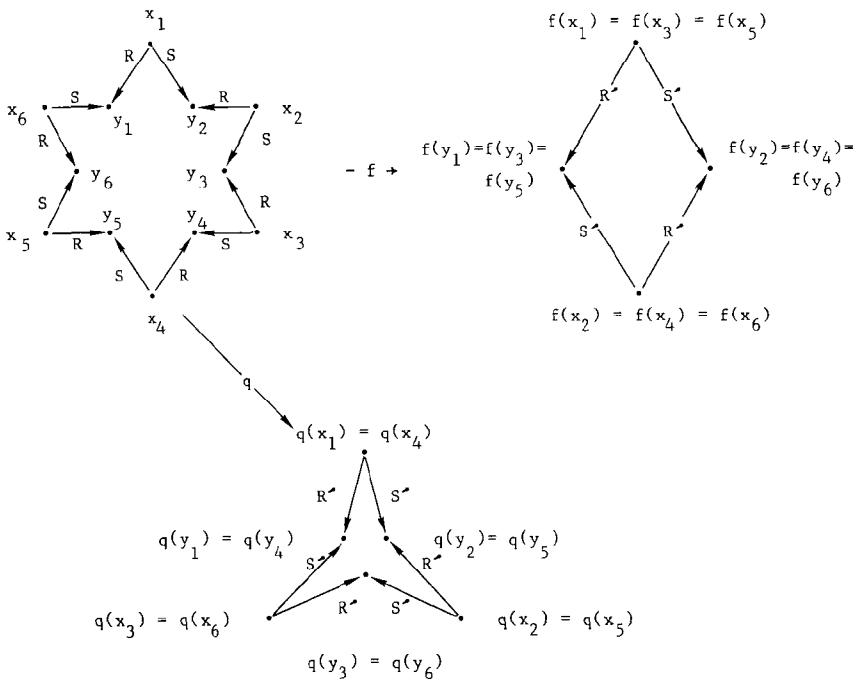
Example 8

Juncture homomorphisms therefore are intermediate to regular and strong homomorphisms. They preserve both properties necessary in the description of roles: multiplexity and composition. Juncture homomorphisms, unlike regular and strong homomorphisms, do not have maximal members, as the following example shows.

Example 8 shows that there is not a simplest unique representation of a role structure. All three networks are multiplex graphs and both f and q are juncture homomorphisms but not strong. Neither of the two image networks can be further reduced while preserving a distinction between the relations.

The implications of the juncture homomorphism for the analysis of role structure are revealing of a fundamental dilemma. If we require unambiguous contents of role relationships, then the simplest reductions of the role positions of actors are not necessarily uniquely determined. This is consistent with Heil and White's (1976) homomorphic approach to role structure, as well as Nadel's more general theoretical rationale for the multiplicity of social structure.

Juncture equivalences can be defined as follows.

Definition 24. Let $\equiv$ be a regular network equivalence, then $\equiv$ is a *juncture network equivalence* if for every $a, b, c, d \in N$, $a \equiv b$ and $c \equiv d$ implies $B_{ac} = B_{bd}$, $B_{ac} = \emptyset$, or $B_{bd} = \emptyset$.

We have now defined a partial order of homomorphisms and their equivalences on networks.
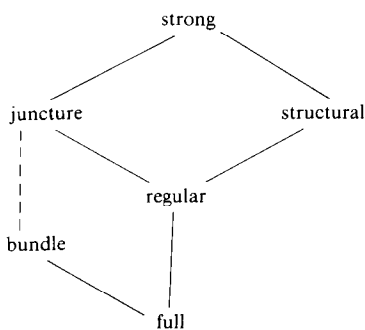
Note: In the case of a multiplex network, a juncture homomorphism is a special case of a bundle homomorphism.

Table 1
Five graph homomorphisms and their properties

Homo-morphism	Image	Induced semigroup homomorphism	Maximal equivalence	Connectivity preserving
Strong	Strong blockmodel	Isomorphism	Yes	Strong
Structural	Structural blockmodel	Isomorphic for irreflexive graph	Yes	Strong
Juncture	Junctural blockmodel	Isomorphism	No	Weak
Regular	Regular blockmodel	Homomorphism	Yes	Weak
Full	Blockmodel	Not necessarily a Homomorphism	Yes	No

Table 1 reviews these homomorphisms and their properties. Three homomorphisms (strong, structural, and regular) have maximal members. Juncture homomorphisms, although lacking the property of having maximal members, have the desirable properties of preserving semigroup structures and multiplexity. The fact that unique maximal members do not occur indicates that "role" may sometimes be analyzed from different and incompatible vantages.

2.5. Network blockmodels and cross-cutting roles

Previous approaches to empirical blocking methods are based on various generalizations of the full homomorphism via a density criterion (Arabie, Boorman and Levitt 1978: 32), or approximation to a structural homomorphism.

The density homomorphism can be defined in the following way. Let N be the network $\langle P, \mathcal{R} \rangle$ and let $f_1: P \to P'$ be an onto map. For the points $a, b \in P'$ let $f_1^{-1}(a) = \{x_1, \ldots, x_{n_a}\}$ and $f_1^{-1}(b) = \{y_1, \ldots, y_{n_b}\}$. Then for each pair $\langle a, b \rangle \in P' \times P'$ and $R \in \mathcal{R}$ we can define the number

$$r_{ab}^R = \frac{\sum_{i}^{n_a} \sum_{j}^{n_b} [x_i R y_j]}{n_a \times n_b} \qquad \text{where } [x_i R y_j] = \begin{cases} 1 & \text{if } \langle x_i, y_j \rangle \in R \\ 0 & \text{if } \langle x_i, y_i \rangle \notin R. \end{cases}$$

Definition 25. Let $N = \langle P, \mathcal{R} \rangle$ be a network and $f_1: P \to P'$ an onto map. For each $R \in \mathcal{R}$ let R' be a relation such that, for each a, b in P', $a\mathcal{R}'b$ that is, a is not related by R' to b, if $r_{ab}^R \leq \alpha$, and $aR'b$ if $r_{ab}^R > \beta$.

The map $f = \langle f_1, f_2 \rangle$ is a *density homomorphism* with a "zeroblock" impurity parameter $\alpha \geqslant 0$ and a "oneblock" impurity parameter $\beta < 1$, if and only if there exist no $a, b \in P'$ such that $\alpha < r_{ab}^R \leq \beta$. We write $\beta \to 1$ to indicate that β is set sufficiently close to 1 so that perfect oneblock density is required.

If $\alpha = \beta$ then any partition of points can be sent via the density homomorphism into an image. Our *full* network homomorphism is the case where $\alpha = \beta = 0$. Breiger, Boorman and Arabie (1975) refer to the case where $\alpha = \beta \approx 0$ as *lean fit*. As α and β move further apart, ruling out intermediate densities, fewer partitions will satisfy the density homomorphism. Where $\alpha = 0$ and $\beta \to 1$, there is only one maximal partition satisfying the homomorphism. This corresponds to our *strong* network homomorphism. Modification of the density parameter r_{ab}^R to exclude reflexive relations corresponds to the case of our *structural* network homomorphism, where again $\alpha = 0$ and $\beta \to 1$. Breiger, Boorman and Arabie refer to the case where $a \approx 0$ and $b \approx 1$ as *fat fit*. Note that there is no way to define regular or juncture homomorphisms via these parameters as density homomorphisms.

To summarize the shortcomings of previous work on blockmodeling:

(1) Structural equivalence yields groupings of points which are related to each other and to all other points in identical ways; network blockmodeling under this equivalence preserves the semigroup of relations on the network, but is too restrictive to capture the more abstract basis of role parallels. This restrictiveness increases for strong equivalence, where equivalent points are related to each other, to themselves, and to all other points in identical ways. The restrictiveness of structural equivalence is relaxed somewhat by "fat fit" approximations (*e.g.*, Breiger, Boorman and Arabie 1975), but these still fail to capture the idea of role parallels, or being related in the same way to equivalent alters.

(2) Full equivalence, even if generalized in terms of a "lean fit" density of exceptions allowed in zeroblocks, is too coarse an equivalence to capture role parallels. If the density cutoff is below the density of the entire graph, the maximal full blockmodel is a single point connected to itself. Structured role regularities in the patterns of connections in the original network must be captured by other criteria.

(3) Bundle equivalence (Mandel and Winship 1979) is an improvement, but is still too coarse to capture positional or role equivalence.

The two criteria for blockmodeling we introduce here are based on definitions of the abstract pattern of role relatedness:

(4) Regular equivalence yields groupings of points in which for every pair of persons in the equivalent position, *if one has a relation with a person in a second position, the other has an identical relation with a counterpart in that position.* This equivalence has two related problems. One is that in networks with multiple relations there may be more than one characteristic multiplex relation (Definition 20) between positions. Because of this, the regular homomorphism does not necessarily preserve the structure of the semigroup of relations defined on the multigraph.

(5) Juncture equivalence is a regular equivalence in which (a) there is no more than one characteristic bundle of relations between positions; and (b) the semigroup of relations defined on the multigraph is isomorphically preserved in the blockmodel image.

The limitation of juncture equivalence is that there is not necessarily a *maximal* juncture equivalence for a given network. Consequently, if juncture equivalence most closely captures the abstract role concept, we may find for a given network that there are a multiplicity of maximal role structures (blockmodels) which do not reduce to a single role system.

This limitation is simply a statement of the obvious fact that roles may *cross-cut* one another in complex ways such that it is impossible to assign every actor a unique role position and then characterize the relations between these positions.

We are thus impelled by this study of homomorphisms towards a formal solution of the problem of how, in a multiplex network, to delimit the distinct cross-cutting role-like structures. This topic will not be examined here.

White and Reitz (1982) present an algorithm which finds a measure of the degree to which nodes in a network are regularly equivalent. Using this algorithm they analyze several different sets of data and give a reduced role structure for each.

3. Semigroup homomorphisms

Lorrain and White (1971), Pattison (1980), and most recently Bona-cich (1982) have explored the relationships between graph homomor-phisms and semigroup homomorphisms. We have shown that under conditions of regularity, network homomorphisms induce semigroup homomorphisms on the semigroup of relations on the graph. In this section, we will explore the converse idea. Semigroup homomorphisms can not be said to induce mappings on the underlying graph in the same fashion that graph homomorphisms induce mappings on the relations. However, semigroup homomorphisms can be termed "com-patible" with graph homomorphisms in the sense that the semigroup homomorphisms can serve as the relational mapping in the ordered pair of mappings which make up a network homomorphism. We will show that in fact certain types of semigroup homomorphisms have this property.

Let $\langle P, S \rangle$ be a network with P the set of points and S a set of binary relations on P. Furthermore, suppose S is a semigroup under the operation of composition. Let $f = \langle f_1, f_2 \rangle$ be a weak full network homomorphism. Then $[R]f_2 = \{Q: f_2(Q) = f_2(R)\}$ is the equivalence class of all relations identified with R by f_2. $\cup[R]f_2$ is the relation consisting of those pairs in any of the relations in $[R]f_2$.

A network homomorphism which collapses only relations and not points is a union of these relations:

Theorem 14. If $N = \langle P, S \rangle$ and $N' = \langle P, S' \rangle$ are networks, $f_1: P \to P$ the identity, and $f_2: S \to S'$ a mapping then $f = \langle f_1, f_2 \rangle$ is a weak full network homomorphism if and only if $f_2(R) = \cup[R]f_2$.

Theorem 14 gives the necessary and sufficient conditions under which a mapping f_2 can be thought of as a network homomorphism. It also gives us a procedure for constructing a network homomorphism which collapses only relations. That is, relations are collapsed simply by forming their union. However, the mapping so constructed does not necessarily preserve composition of relations.

Suppose S is a semigroup of relations on a network. Bonacich (1982) poses the problem of how to consider semigroup homomorphisms on a network in terms of unions of relations which preserve composition. The problem can be stated as follows. Given a network $\langle P, S \rangle$ and a

semigroup homomorphism f: $\langle S, \circ \rangle \to \langle S', \circ \rangle$, under what conditions can f be represented as the relational map from the network $\langle P, S \rangle$ to a network $\langle P, S' \rangle$? Note that any solution to Bonacich's problem must first show under what conditions an arbitrary semigroup image of a semigroup of binary relations can be represented as another semigroup of binary relations. Ideally, some such conditions can be specified with as little reference as possible to the underlying network.

Given $f = \langle f_1, f_2 \rangle$: $\langle P, S \rangle \to \langle P, S' \rangle$, a network homomorphism, we will develop necessary and sufficient conditions for f_2 and S such that f_2: $\langle S, \circ \rangle \to \langle S', \circ \rangle$ is a semigroup homomorphism. Thus given the semigroup $\langle S, \circ \rangle$ and a semigroup homomorphism f_2: $\langle S, \circ \rangle \to \langle S', * \rangle$, where $\langle S', * \rangle$ is any semigroup image of $\langle S, \circ \rangle$, we will have necessary conditions on S and f_2 such that $\langle S', * \rangle$ is isomorphic to a semigroup of binary relations $\langle S', \circ \rangle$ on a set P and $\langle i, f_2 \rangle$ is a network homomorphism with i the identity.

The following example illustrates the problem. Given the network $N = \langle P, S \rangle$ as illustrated below, the composition table for $\langle S, \circ \rangle$ can be derived as follows.

o	∅	Q	R	T	U
∅	∅	∅	∅	∅	∅
Q	∅	∅	T	∅	∅
R	∅	∅	∅	∅	∅
T	∅	∅	∅	∅	∅
U	∅	∅	∅	∅	∅
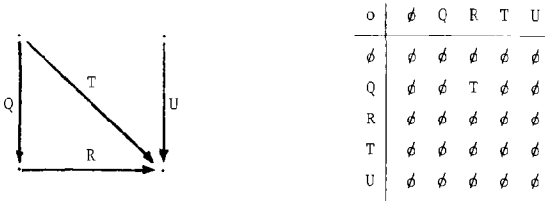
Example 9

Note that a homomorphic image of $\langle S, \circ \rangle$ can be formed by identifying the relations T and U. This new semigroup $\langle S', * \rangle$ has the following composition table:

*	∅	Q'	R'	T'
∅	∅	∅	∅	∅
Q'	∅	∅	T'	∅
R'	∅	∅	∅	∅
T'	∅	∅	∅	∅

Theorem 14 tells us that if T and U are to be identified as part of a weak full network homomorphism, the image T' of T and U must be

the union of T and U, which gives the following network image of N:
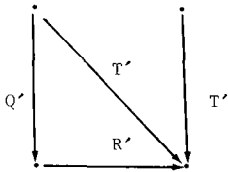
Note, however, that in this case $Q' \circ R'$ is not equal to T' but only a subset of T'. Stated another way, $[Q]f_2 \circ [R]f_2 \subset \cup[Q \circ R]f_2$. In order for f_2 to be a semigroup homomorphism equality must hold. This motivates the following theorem for the necessary and sufficient conditions under which a relational map in a network homomorphism is a semigroup homomorphism.

Theorem 15. Let $f = \langle f_1, f_2 \rangle : \langle P, S \rangle \rightarrow \langle P, S' \rangle$ be a weak full network homomorphism such that f_1 is the identity mapping. Then f_2 preserves composition if and only if $\cup[R_1]f_2 * \cup[R_2]f_2 = \cup[R_1 \circ R_2]f_2$.

Suppose S is a semigroup of binary relations under composition. Then under the conditions stated in Theorem 15, f_2 is a semigroup homomorphism. From semigroup theory, we know that the compositions of semigroup homomorphisms are semigroup homomorphisms.

Theorem 15 characterizes the general problem posed by Bonacich. It simply states that every relational mapping embedded in a network homomorphism, even if it appears to be a semigroup homomorphism of a semigroup composition table, must in fact be checked to insure that it preserves composition on the network of relations. Since the weak full network homomorphism creates images of relations in terms of the unions of equated relations, the necessary and sufficient conditions to insure the preservation of composition consists of actually checking the compositions of these unions to insure, for all R_1, R_2 in the set S of relations, that $f_2(R_1) \circ f_2(R_1) = f_2(R_1 \circ R_2)$.

The more specific problem posed by Bonacich is whether there are characteristics of S and f_2 which, independent of the underlying network, insure that a relation mapping in a network homomorphism will preserve composition.

We are particularly interested in highly interpretable semigroup homomorphisms. One reason for collapsing relations in a semigroup, for example, is that they have the same products. This is analogous to strong equivalence in the context of collapsing points in a graph when they have identical linkages. In the relation context, having the same products means that two relations generate identical paths of length two or more in the network. Here, then, we begin with defining a strong relational mapping and its induced equivalence.

Definition 26.   Let $f: S \to S'$ be a mapping of relations and $\langle S, \circ \rangle$ a semigroup of relations under composition. Then f is *strong* if for all T, R_1, R_2 in S, $f(R_1) = f(R_2)$ if and only if $R_1 \circ T = R_2 \circ T$ and $T \circ R_1 = T \circ R_2$.

Definition 27.   Let R_1 and R_2 be elements of S. Then R_1 and R_2 are *strong equivalent* (denoted by $R_1 \equiv R_2$) if for all T in S, $R_1 \circ T = R_2 \circ T$ and $T \circ R_1 = T \circ R_2$.

It is clear from the definition that strong equivalence is indeed an equivalence relation. While it satisfies a condition stronger than congruence on the semigroup $\langle S, \circ \rangle$, namely that $R_1 \equiv R_2$ and $Q_1 \equiv Q_2$ implies $R_1 \circ Q_1 = R_2 \circ Q_2$, the semigroup $S/\equiv$ is not necessarily a semigroup of binary relations, as seen in Example 9.

The proof of the following theorem follows directly from these definitions.

Theorem 16.   The equivalence on S induced by a strong mapping is a strong equivalence and conversely every strong equivalence is induced by some strong mapping.

It is not the case that the composition of two strong mappings is a strong mapping, as shown by the following example.

$\circ$	R	S	T
R	R	R	R
S	R	R	R
T	R	R	S

$- g_1 \to$

$\circ$	U	T
U	U	U
T	U	U

$- g_2 \to$

$\circ$	V
V	V

Example 10

where $g_1(R) = g_1(S) = U$, and $g_2(U) = g_2(T) = V$. Both g_1 and g_2 are

strong but $g_2 \circ g_1$ is not. Because of the interpretability of strong mappings, the following definition is useful.

Definition 28. A relational mapping which can be written as the composition of two or more strong relational mappings is *quasistrong*.

While we will not prove it here as a theorem, both strong and quasi-strong relational mappings of semigroups are semigroup homomor-phisms (as in Example 10), but not with respect to relational unions and their compositions. That is, strong and quasistrong mappings cannot necessarily be the relational map in a network homomorphism and preserve composition. We now show the condition under which they do preserve composition.

We have defined the strong relational mapping and its congruence without reference to networks. Lemma 4 gives us the necessary and sufficient conditions in order that the relational part of a network mapping preserves compositions. Under the condition that the rela-tional map be strong we can prove a stronger theorem.

Theorem 17. Let $f = \langle f_1, f_2 \rangle : \langle P, S \rangle \rightarrow \langle P, S' \rangle$ be a weak full net-work homomorphism, f_1 the identity, f_2 a strong mapping, and $\langle S, \circ \rangle$ a semigroup. Then f_2 is a semigroup homomorphism if and only if $f_2(T) = f_2(R_1 \circ R_2)$ implies $T \subseteq R_1 \circ R_2$ for all T, R_1, R_2 in S.

This is a strong result: it allows us to identify strong relational mappings from the semigroup composition table (by equality of prod-ucts) and check whether the unions of equated relations preserve composition by inspection of the lattice of inclusions among elements of S, *without* further reference to the underlying network.

Reductions of this form can be chained to yield further semigroup homomorphisms on a network. At each step a new inclusion lattice among relational elements is computed by taking unions of relations equated in the previous step, and checking each new pair of relations for set inclusion. In a chain of such mappings, the relational mapping becomes quasistrong.

The quasistrong homomorphism of a network allows the following interpretation. Starting with the original semigroup of relations on a network, we equate all strongly equivalent relations as role "substitut-able" in generating identical paths in the network of length two or

more. Further relations may be equated on this basis in the image network. The largest strong reduction at each step is unique, and the largest quasistrong reduction is unique. It defines a hierarchy of levels in which relations are equated or role "substitutable." If the strong relational mappings in Example 10 were composition preserving in a network homomorphism, we would have the role "substitutability" hierarchy:
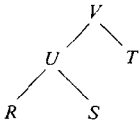
The relations at the bottom were equated in the first strong relation mapping, and those in the middle equated in the next strong mapping.

There is a second related condition for a semigroup homomorphism of relations on a network which is highly interpretable. Like the previous one, it makes use of inclusions between relations. The second condition is more directly motivated by Boorman and White's (1976) idea of equating relations linked by inclusion.

Definition 29.   Let $\langle S, \circ \rangle$ be a semigroup of binary relations and f a map on S. Then f is an *inclusion* map if for all R_1, R_2 in S, $f(R_1) = f(R_2)$ implies $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$.

Unlike the strong mapping, there is not necessarily a maximal inclusion map of a semigroup of relations. For example, there may be two inclusions between three relations, where $P \subseteq R$ and $Q \subseteq R$. Only two of the three (either P and R or Q and R) can be equated since the other two are not linked by inclusion. Congruence does not necessarily hold, and f is not necessarily a semigroup homomorphism.

If $f = \langle f_1, f_2 \rangle$ is a weak full network homomorphism in which f_1 is the identity and f_2 an inclusion map, it is easy to see that among the relations identified by f_2 there exists a maximal relation. This is because all equated elements are linked by inclusion, which defines a total ordering of elements within each equivalence set. Therefore $f_2(R) = \max[R]f_2$.

The following characterizes the conditions under which an inclusion map is a semigroup homomorphism.

Theorem 18. Let $f = \langle f_1, f_2 \rangle : \langle P, S \rangle \rightarrow \langle P, S' \rangle$ where f_1 is the identity map and f_2 is an inclusion mapping. Then f_2 preserves composition if and only if

(i) $f_2(R_1) = f_2(R_2)$ implies $f_2(T \circ R_1) = f_2(T \circ R_2)$ and $f_2(R_1 \circ T) = f_2(R_2 \circ T)$; and

(ii) $f_2(T) = f_2(R_1 \circ R_2)$ implies $T \subseteq \max[R_1]f_2 \circ \max[R_2]f_2$; for all T, R_1, R_2 in S.

This is the same type of strong result we obtained for checking whether a strong relation mapping qualifies as a semigroup homomorphism on a network of relations. Here, we can identify inclusion mappings which preserve composition from the semigroup composition tables plus the inclusion lattice of the relations.

Semigroup homomorphisms of the relations on a network defined by inclusion mappings can, like strong mappings, be composed to yield further semigroup homomorphisms. However, since inclusion is transitive, the composition of two inclusion mappings is also an inclusion mapping, and can be identified in a single step from the original semigroup composition table of relations on the network and the inclusion lattice.

Our first two approaches come close to the intuitions of Boorman and White (1976), who include in their description of the "role structure" of a social network both the semigroup table of compositions of relations and the inclusion lattice among the relations. Their concept of "role structure" is thus sufficient to characterize the strong and inclusion semigroup homomorphisms of social networks. These semigroup homomorphisms are highly interpretable in terms of the "substitutability" of relations, and are analogous to the criteria we used earlier to characterize the equivalence of points in the network.

Homomorphisms on the semigroup of all binary relations on a set which are also union and symmetry preserving have been well characterized by Clifford and Miller (1970) and Magill (1966). Inclusion homomorphisms on a subsemigroup of the set of binary relations on a set N can be easily extended to the set of all relations (via lattices). There are, however, no strongly equivalent relations in the semigroup of all binary relations on a set. The utility of quasistrong homomorphisms is thus restricted to subsemigroups of the set of all binary relations on a set.

The definitions and theorems in this final section of the paper

provide basic tools for the structural analysis of semigroup homomor-
phisms in social networks. One line of investigation consists of a
semigroup reduction by a chain of strong semigroup homomorphisms.
Another consists of reduction via the inclusion homomorphisms. A
third consists of alternating the two types of semigroup reductions. The
definitions and theorems in the first part of the paper provide basic
tools for homomorphic reductions of points in a network which pre-
serve aspects of semigroup and graph structure. Together these methods
provide a powerful basis for extending blockmodel analyses of the role
structure of social networks.

The representation we have chosen of network homomorphisms is a
powerful one. It enables us to prove a number of significant theorems
regarding the role structure of social networks, to illuminate a number
of natural-language intuitions concerning social roles, and to yield
deeper insights into the type of role structure analysis developed by
Lorrain and White (1971), White, Boorman and Breiger (1976), and
Boorman and White (1976). It allows us to integrate the analysis of
graph homomorphisms and the analysis of semigroup homomorphisms
of binary relations under composition. Our results open the door to
further mathematical study of the relation between semigroup and
graph homomorphisms on networks. For the analyst of social networks,
our theorems represent a considerable practical advance in the methods
for deriving homomorphic images of social networks. Such methods
provide the formal basis for the network analysis of complex social
structures.

Appendix

Proposition A.   Let $G = \langle P, R \rangle$ be a graph and $f: P \to P'$ be an onto
map for some set P'. If R' is the relation on P' induced by R and f, and
$G' = \langle P', R' \rangle$, then the map $f: G \to G'$ is a full graph homomorphism.

Proof.   If aRb for $a, b \in P$ then by definition $f(a)R'f(b)$. If $xR'y$ for
some $x, y \in P'$ then $\langle x, y \rangle \in R'$. But $R' = \{\langle f(a), f(b) \rangle : a, b \in P$ and
$\langle a, b \rangle \in R\}$. Therefore $\langle x, y \rangle = \langle f(a), f(b) \rangle$ for some $a, b \in P$ where
$\langle a, b \rangle \in R$ so $x = f(a)$, $y = f(b)$ and aRb. Therefore f is a full graph
homomorphism.

Theorem 1. If $f: G \to G'$ then
(i) f is a strong homomorphism implies f is a structural homomorphism;
(ii) f is a structural homomorphism implies f is a regular homomorphism.

Proof.
(i) The proof of (i) follows directly from the definition of strong and structural homomorphism.
(ii) Let $f: G \to G'$ be a structural graph homomorphism and suppose $f(a)R'f(b)$. If $a \neq b$ then aRb so let $c = a$ and $d = b$. $f(c) = f(a)$ and cRb, similarly $f(d) = f(b)$ and aRd. If $a = b$ then $f(a)R'f(a)$. Since f is a full graph homomorphism there exist $x, y \in P$ such that $f(x) = f(a)$, $f(y) = f(a)$ and xRy. If both x and y are equal to a then let $c = d = a$. If $y \neq a$, then $f(y)R'f(a)$ and by the definition of f being structural yRa so let $c = y$. Similarly $f(a)R'f(y)$ so aRy, $d = y$. In an identical fashion, if $x \neq a$, let $c = d = x$. In each of these cases given $f(a)R'f(b)$ we have shown the existence of some c and $d \in P$ such that $f(c) = f(a)$, $f(d) = f(b)$, cRb, and aRd.

Theorem 2A. If $\equiv_f$ is an equivalence relation on the set P induced by a full homomorphism f, then f is a strong graph homomorphism $f: G \to G'$ if and only if for all $a, b, c \in P$,
$a \equiv_f b$ implies   (i) aRb if and only if bRa;
                       (ii) aRc if and only if bRc; and
                       (iii) cRa if and only if cRb.

Proof. Let $f: G \to G'$ be a strong homomorphism and let $\equiv_f$ be the equivalence relation induced by f. If $a \equiv_f b$ then $f(a) = f(b)$. Now if aRb, $f(a)R'f(b)$, and $f(b)R'f(a)$. Since f is strong, bRa. By symmetry if bRa then aRb. So $a \equiv_f b$ implies (i). Similarly if $a \equiv_f b$ and aRc then $f(a)R'f(c)$ but since $f(a) = f(b)$, $f(b)R'f(c)$ and bRc. Similarly bRc implies aRc. Therefore $a \equiv_f b$ implies (ii). Finally, if $a \equiv_f b$ and cRa then $f(c)R'f(a)$ and since $f(a) = f(b)$, $f(c)R'f(b)$ so cRb. Similarly cRb implies cRa. So $a \equiv_f b$ implies (iii). An argument similar to the reverse of the above proves the converse.

Theorem 2B. If $\equiv_f$ is an equivalence relation on a set P induced by a full homomorphism f, then f is a structural graph homomorphism if

and only if for all a, b, c in P, where $c \neq a$ and $c \neq b$,

$a \equiv_f b$ implies   (i) aRb if and only if bRa;

    (ii) aRc if and only if bRc;

    (iii) cRa if and only if cRb; and

    (iv) aRa implies aRb.

Proof. Let $f: G \to G'$ be a structural graph homomorphism; let $a \equiv_f b$. If aRb then $f(a)R'f(b)$. Since $a \equiv_f b$, $f(a) = f(b)$ and $f(b)R'f(a)$. If $a = b$, aRb implies bRa immediately. However, if $a \neq b$, $f(b)R'f(a)$ implies by the definition of structural homomorphism that bRa. Similarly if bRa then aRb. So $a \equiv_f b$ implies (i). If aRc, $a \neq c$, $b \neq c$ and $a \equiv_f b$ implies (ii). An almost identical argument proves that $a \equiv_f b$ implies (iii). If aRa and $a = b$ then aRb. If aRa and $a \neq b$ then $f(a)R'f(a)$ and $f(a)R'f(b)$. Since f is structural, aRb, so (iv) holds and $\equiv_f$ is structural. Conversely, suppose f is full, $\equiv_f$ is induced by f and $\equiv_f$ is structural. Then if $a \neq b$ and $f(a)R'f(b)$ there exists c, $d \in P$ such that $f(c) = f(a)$, $f(d) = f(b)$ and cRd. There are essentially 5 distinct cases to consider: (1) a, b, c, and d are all distinct; (2) $a = c$, $b \neq d$, and $c \neq d$; (3) $a = d$, $b \neq c$ and $c \neq d$; (4) $c = d$, $a \neq c$ and $b \neq c$; (5) $a = d$ and $b = c$. The proof for case (1) is identical to that for a strong homomorphism. The proofs for cases (2) through (5) are similar and only the proof for (4) will be presented. If cRd then cRc since $c = d$. By condition (iv) cRa and by condition (i) aRc. This in turn implies aRd which by condition (iii) implies aRb.   Q.E.D.

Theorem 2C. If $\equiv_f$ is an equivalence relation induced by a full homomorphism f, then f is induced by a regular homomorphism if and only if for all a, b, $c \in P$.

$a \equiv_f b$ implies

 (i) aRc implies there exists $d \in P$ such that bRd and $d \equiv_f c$; and

 (ii) cRa implies there exists $d \in P$ such that dRb and $d \equiv_f c$.

Proof. If f is a regular homomorphism, $a \equiv_f b$, and aRc then $f(a) = f(b)$ and $f(b)R'f(c)$. Since f is regular there exists $d \in P$ such that $f(d) = f(c)$ and bRd. Since $f(d) = f(c)$, $d \equiv_f c$. Therefore $a \equiv_f b$ implies (i). If $a \equiv_f b$ and cRa then $f(c)R'f(b)$ and there exists $d \in P$ such that $f(d) = f(c)$ and dRb. Again $d \equiv_f c$. Therefore $a \equiv_f b$ implies (ii). An argument similar to the reverse of the above proves the converse.

Theorem 3A. The collection of all strong equivalences on a graph has a maximal element.

Proof. Let E be the collection of all strong equivalence relations on P and let
$\equiv_M = \{\langle a, b \rangle : a, b \in P$ which satisfy

(i) aRb if and only if bRa;

(ii) aRc if and only if bRc; and

(iii) cRa if and only if cRb for all $c \in P\}$.

We must first show that $\equiv_M$ is indeed an equivalence relation. If $a \in P$, aRa if and only if aRa so condition (i) holds. aRc if and only if aRc so condition (ii) holds. Similarly condition (iii) holds and $a \equiv_M a$ for all $a \in P$ and $\equiv_M$ is reflexive. If $a \equiv_M b$ then the symmetry of conditions (i), (ii), and (iii) implies that $b \equiv_M a$ so $\equiv_M$ is symmetric. Now suppose $a \equiv_M b$ and $b \equiv_M d$. By the conditions, aRd if and only if bRd if and only if dRb if and only if dRa so condition (i) holds for the pair $\langle a, d \rangle$. By condition (ii) aRc if and only if bRc if and only if dRc so condition (ii) holds for the pair $\langle a, d \rangle$. A similar argument shows that condition (iii) holds for $\langle a, d \rangle$ and therefore $a \equiv_M d$ and $\equiv_M$ is transitive. Therefore $\equiv_M$ is a strong equivalence relation. All that remains to be shown is that $\equiv_M$ is maximal. Let $\equiv$ be a strong equivalence relation on P. By definition of what a strong equivalence relation is, $a \equiv b$ implies that conditions (i)–(iii) are satisfied and $a \equiv_M b$. Therefore $\equiv_M$ is maximal.   Q.E.D.

Theorem 3B. The collection of all structural equivalences on a graph has a maximal element.

Proof. The proof is similar to the proof of 3A except that conditions (ii) and (iii) have the added condition that $a \neq c \neq b$. Every conclusion still follows.

Theorem 3C. The collection of all regular equivalences on a graph has a maximal element.

Proof. Let $\equiv_1 = \{\langle a, b \rangle \in P \times P\}$,
$\equiv_2 = \{\langle a, b \rangle \in P \times P:$

(i) aRc implies there exists $d \in P$ such that bRd and $d \equiv_1 c$;

(ii) bRc implies there exists $d \in P$ such that aRd and $d \equiv_1 c$;

(iii) cRa implies there exists $d \in P$ such that dRb and $d \equiv_1 c$; and

(iv) cRb implies there exists $d \in P$ such that dRa and $d \equiv_1 c$}.

Similarly define $\equiv_3$ in terms of $\equiv_2$ as $\equiv_2$ has been defined in terms of $\equiv_1$. Continue in this fashion defining recursively $\equiv_{k+1}$ in terms of $\equiv_k$. We first show that each $\equiv_k$ is an equivalence relation. Certainly $\equiv_1$ is an equivalence relation. Assume $\equiv_k$ is also. Let $a \in P$, then aRc implies aRc and $c \equiv_k c$ since $\equiv_k$ is an equivalence relation. Similarly conditions (ii)–(iv) hold and $a \equiv_{k+1} a$ and $\equiv_{k+1}$ is reflexive. Suppose $a \equiv_{k+1} b$ then by the symmetry of the conditions (i)–(iv) $b \equiv_{k+1} a$ and $\equiv_{k+1}$ is symmetric. Suppose $a \equiv_{k+1} b$ and $b \equiv_{k+1} x$, then aRc implies there exists $d \in P$ such that bRd and $d \equiv_k c$. But if bRd, then there exists an $e \in P$ such that xRe and $e \equiv_k c$. But since $\equiv_k$ is an equivalence relation $e \equiv_k c$, so xRe and $e \equiv_k c$ so the pair $\langle a, x \rangle$ satisfies condition (i). Similarly $\langle a, x \rangle$ satisfies conditions (ii)–(iv) and $a \equiv_{k+1} x$ so $\equiv_{k+1}$ is transitive. So by assuming $\equiv_k$ is an equivalence relation, we have shown that $\equiv_{k+1}$ is an equivalence relation. By the axiom of mathematical induction $\equiv_k$ is an equivalence relation for all k.

Next we show that $\equiv_k \supseteq \equiv_{k+1}$ for all i. Certainly $\equiv_2 \subseteq \equiv_1$. Assume $\equiv_k \subseteq \equiv_{k-1}$ and let $\langle a, b \rangle \in \equiv_{k+1}$. Suppose aRc, then there exists $d \in P$ such that bRd and $d \equiv_k c$. But $\equiv_k \subseteq \equiv_{k-1}$ so $d \equiv_{k-1} c$. That means the pair $\langle a, b \rangle$ satisfies condition (i) for $\equiv_k$. In a similar fashion, $\langle a, b \rangle$ satisfies conditions (ii)–(iv) for $\equiv_k$ and $\langle a, b \rangle \in \equiv_k$. So $\equiv_k \supseteq \equiv_{k+1}$. By the axiom of induction $\equiv_k \supseteq \equiv_{k+1}$ for all k. Since P is finite there must be an N such that $\equiv_N = \equiv_{N-1}$. This means that $\equiv_N$ is defined in terms of itself and therefore satisfies the properties of a regular equivalence relation.

All that remains to be shown is that $\equiv_N$ is maximal. Suppose $\equiv$ is a regular equivalence relation. Clearly $\equiv \subseteq \equiv_1$. Suppose $\equiv \subseteq \equiv_k$, let $a \equiv b$, then if aRc, by the definition of regular equivalence relations there exists $d \in P$ such that bRd and $d \equiv c$. But since $\equiv \subseteq \equiv_k$, $d \equiv_k c$, so the pair $\langle a, b \rangle$ satisfies condition (i) of $\equiv_{k+1}$. Similarly $\langle a, b \rangle$ satisfies condition (ii)–(iv) and $\langle a, b \rangle \in \equiv_{k+1}$. So $\equiv \subseteq \equiv_{k+1}$ and hence by induction $\equiv \subseteq \equiv_k$ for all k. Therefore $\equiv \subseteq \equiv_N$ and $\equiv_N$ is the maximal regular equivalence relation.   Q.E.D.

The following lemma although not in the text, is useful in the proof of the following theorems.

Lemma. If $f: G \to G'$ is a regular graph homomorphism and R' is the relation induced by f and R then $(R^i)' = (R')^i$.

Proof. Let $\langle x, y \rangle \in (R^i)'$, then there exists $\langle a, b \rangle \in R^i$ such that $\langle f(a), f(b) \rangle = \langle x, y \rangle$. If $\langle a, b \rangle \in R^i$, there exists $c_1, c_2, \ldots, c_{i-1}$ such that $\langle a, c_1 \rangle, \langle c_1, c_2 \rangle, \ldots, \langle c_{i-1}, b \rangle$ are all members of R and $\langle f(a), f(c_1) \rangle, \langle f(c_1), f(c_2) \rangle, \ldots, \langle f(c_{i-1}), f(b) \rangle$ are all members of R'. But this implies that $\langle f(a), f(b) \rangle = \langle x, y \rangle \in (R')^i$. So $(R^i)' \subseteq (R')^i$.

Now suppose $\langle x, y \rangle \in (R')^i$, then there exists $w_1, \ldots, w_{i-1}$ such that $\langle x, w_1 \rangle, \langle w_1, w_2 \rangle, \ldots, \langle w_{i-1}, y \rangle$ are all members of R'. Since f is full, there exists $\langle a, c_1 \rangle \in R$ such that $\langle f(a), f(c_1) \rangle = \langle x, w_1 \rangle$. So $\langle f(c_1), w_2 \rangle \in R'$ and since f is regular there exists c_2 such that $f(c_2) = w_2$ and $\langle c_1, c_2 \rangle \in R$. In similar fashion there exists $c_3, \ldots, c_{i-1}, b$ such that $\langle c_l, c_{l+1} \rangle \in R$ and $f(c_l) = w_l$ and $\langle c_{i-1}, b \rangle \in R$ with $f(b) = y$. Therefore $\langle a, b \rangle \in R^i$ and $\langle f(a), f(b) \rangle = \langle x, y \rangle \in (R^i)'$. We have then $(R^i)' \supseteq (R')^i$.   Q.E.D.

Theorem 4. If $f\colon G \to G'$ is a regular, structural, or strong graph homomorphism then f is regular, structural or strong respectively for any relation in S. That is $f\colon \langle P, Q \rangle \to \langle P', Q' \rangle$ is regular, structural, or strong respectively for any $Q \in S$.

Proof. In the case that f is regular, the theorem holds trivially for $Q = R^1 = R$. Suppose it holds for $Q = R^i$. Consider

$$f\colon \langle P, R^{i+1} \rangle \to \langle P', (R')^{i+1} \rangle.$$

If $f(a)(R')^{i+1}f(b)$, then there is some $x \in P'$ such that $f(a)(R')^i x$ and $xR'f(b)$. Since f is "onto", there exists $c \in P$ such that $f(c) = x$, so $f(a)(R')^i f(c)$. Since f is regular with respect to R^i there exists $e \in P$ such that $f(c) = f(e)$ and $aR^i e$. Similarly there exists $s \in P$ such that $f(s) = f(b)$, and eRs. Note that $aR^i e$ and eRs so $aR^{i+1}s$ and $f(s) = f(b)$ so the first half of the condition for regularity holds. Since $f(c)R'f(b)$, there exists $u \in P$ such that $f(u) = f(c)$ and uRb. Since $f(u) = f(c)$, $f(a)(R')^i f(u)$, so by the induction hypothesis there exists w such that $f(w) = f(a)$ and $wR^i u$. Since $wR^i u$ and uRb we have $wR^{i+1}b$. Thus by the axiom of induction f is regular on R^i for all i. In the case that f is strong with respect to $\langle P, R \rangle$ the proof is less involved. Again for $Q = R'$ the theorem holds. Suppose it holds for $Q = R^i$. If $f(a)(R')^{i+1}f(b)$, then $f(a)(R^{i+1})'f(b)$, whence $aR^{i+1}b$ and f is strong with respect to R^{i+1}. By induction f is strong with respect to R^i for all i.

The proof for the case that f is structural follows exactly as for f strong except that the condition $a \neq b$ must be added.

Q.E.D.

Theorem 5. If $f\colon G \to G'$ is a regular graph homomorphism then $\hat{f}\colon S \to S'$ is a semigroup homomorphism. That is $\hat{f}(Q_1 \circ Q_2) = \hat{f}(Q_1) \circ \hat{f}(Q_2)$.

Proof. $f\colon G \to G'$ is a regular homomorphism. Let $Q_1 = R^i$ and $Q_2 = R^j$; then $Q_1 \circ Q_2 = R^i \circ R^j = R^{i+j}$ so $\hat{f}(Q \circ Q_2) = \hat{f}(R^{i+j}) = (R^{i+j})' = (R')^{i+j} = (R')^i \circ (R')^j = (R^i)' \circ (R^j)' = \hat{f}(R^i) \circ \hat{f}(R^j) = \hat{f}(Q_1) \circ \hat{f}(Q_2)$.

Q.E.D.

Theorem 6. If $f\colon G \to G'$ is a strong homomorphism then $\hat{f}\colon S \to S'$ is a semigroup isomorphism.

Proof. By Theorem 5, $\hat{f}$ is a semigroup homomorphism. Suppose $\hat{f}(Q_1) = \hat{f}(Q_2)$. Then for all pairs $\langle a, b \rangle \in P \times P$, $\langle f(a), f(b) \rangle \in \hat{f}(Q_1)$ if and only if $\langle f(a), f(b) \rangle \in \hat{f}(Q_2)$. But $\langle f(a), f(b) \rangle \in \hat{f}(Q_1)$ if and only if $\langle a, b \rangle \in Q_1$. Similarly $\langle f(a), f(b) \rangle \in \hat{f}(Q_2)$ if and only if $\langle a, b \rangle \in Q_2$. So $\langle a, b \rangle \in Q_1$ if and only if $\langle a, b \rangle \in Q_2$. Therefore $Q_1 = Q_2$ and $\hat{f}$ is an isomorphism.

Q.E.D.

Theorem 7. If $f\colon G \to G'$ is a structural homomorphism and G an acyclic graph then $\hat{f}\colon S \to S'$ is an isomorphism.

Proof. The proof is identical with that of Theorem 6 with the added condition that $a \neq b$. With this condition it is sufficient to hypothesize that $\langle a, a \rangle \notin R^i$ for all i. The conclusion that $Q_1 = Q_2$ follows.

Lemma 2. If $G = \langle P, \equiv_A \rangle$ is the graph of an attribute equivalence relation $\equiv_A$, then $\equiv_A$ is also the equivalence induced by the largest regular, structural, or strong homomorphism of the graph.

Proof. Define $P' = \{0, 1\}$ and let $f\colon P \to P'$ be defined by

$$f(a) = \begin{cases} 1 & a \in A \\ 0 & a \notin A \end{cases}.$$

Then f is a strong homomorphism from $\langle P, \equiv_A \rangle$ to $\langle P', = \rangle$ where

$\iota = 1$, $0 \neq 0$, $0 \neq 1$. $\equiv_A$ is also the equivalence induced by f and is clearly maximal.

Lemma 3. If $G = \langle P, I_A \rangle$ is the graph of a class identity I_A, defined by the class attribute A, then the attribute equivalence $\equiv_A$ is also the equivalence induced by the largest regular homomorphism of the graph.

Proof. The proof is identical to that of Lemma 2 except f is no longer strong but only regular.

Theorem 8. If f is a regular network homomorphism, then f is connectivity preserving.

Proof. Let $x_1, \ldots, x_{n+1} \in P$ be a sequence of points and $R_1, \ldots, R_n$ a set of relations on P such that $f_1(x_1) f_2(R_1) f_1(y_2) \ldots$ $f_1(x_n) f_2(R_n) f_1(x_{n+1})$. Since f is regular, there exists y_2 such that $f_1(y_2) = f_2(x_2)$, and $x_1 R_1 y_2$. Since $f_1(y_2) f_2(R_2) f_1(x_3)$ there exists y_3 such that $f(y_3) = f(x_3)$ and $y_2 R_2 y_3$. Continue in like manner establishing $y_2, y_3, \ldots, y_{n+1}$ with $f_1(x_2) = f_1(y_2), \ldots, f_1(x_{n+1}) = f_1(y_{n+1})$ and $x_1 R_1 y_2 R_2 \ldots R_n y_{n+1}$. Similarly since $f_1(x_n) f_2(R_n) f_1(x_{n+1})$, there exists z_n such that $f_1(x_n) = f_1(z_n)$ and $z_n R_n x_{n+1}$. Since $f_1(x_n) = f_1(z_n)$, $f_1(x_{n-1}) f_2(R_{n-1}) f_1(z_n)$ and there exists a $z_{n-1} \in P$ such that $f_1(z_{n-1}) = f_1(x_{n-1})$ and $z_{n-1} R_{n-1} z_n$. In this manner, we construct $z_1, z_2, \ldots, z_n$ such that $f(z_i) = f(x_i)$ for $1 \leq i \leq n$ and $z_1 R_1 z_2 R_2 \ldots z_n R_n x_{n+1}$. Therefore f is connectivity preserving.

Theorem 9. If f is a structural network homomorphism then f is strongly connectivity preserving.

Proof. The proof follows directly from the definition of structural graph homomorphisms.

Theorem 10. If $N = \langle P, \mathfrak{R} \rangle$ is a network, $C = \langle P, \mathfrak{M} \rangle$ the multiplex graph derived from it, and $f: C \to C' = \langle P', \mathfrak{M}' \rangle$ a full network homomorphism then f induces a full network homomorphism on N and
(i) if f is regular the induced homomorphism is regular;
(ii) if f is strong the induced homomorphism is strong.

Proof. Let $f: C \to C'$ be the ordered pair $\langle f_1, f_2 \rangle$ where $f_1: P \to P'$ and $f_2: \mathfrak{M} \to \mathfrak{M}'$. Let $R \in \mathfrak{R}$ and define $f_2^*(R) = \{\langle f_1(a), f_1(b) \rangle: \langle a, b \rangle$

$\in R$. Let $\mathfrak{R}' = \{f_2^*(\dot{R}): R \in \mathfrak{R}\}$ then the ordered pair $\langle f_1, f_2^* \rangle$ is a full network homomorphism $f^*: N \to N'$ where $N' = \langle P', \mathfrak{R}' \rangle$. Suppose $f_1(a)f_2^*(R)f_1(b)$ for some $a, b \in P$ and $R \in \mathfrak{R}$. Since $f^* = \langle f_1, f_2^* \rangle$ is full, there exist $x, y \in P$ such that $f_1(a) = f_1(x), f_1(b) = f_1(y)$, and xRy. This implies that $R \in B_{xy} \neq \emptyset$. Let $M = \{\langle c, d \rangle : B_{cd} = B_{xy}\}$ then xMy and $f_1(x)f_2(M)f_1(y)$ which in turn implies $f_1(a)f_2(M)f_1(b)$.

(i) If we assume that f is regular, there exist $c, d \in P$ such that $f_1(c) = f_1(a), f_1(d) = f_1(b)$, cMb and aMd. By the definition of M, $B_{cb} = B_{ad} = B_{xy}$, whence cRb, aRd, and f^* is regular.

(ii) If we assume that f is strong, then aMb, $B_{ab} = B_{xy}$, aRb, and f^* is strong.

Theorem 11a. Every strong network homomorphism is a juncture network homomorphism.

Proof. Assume $f: N \to N'$ is strong and there exists $a, b, c, d \in P$ such that $f_1(a) = f_1(c)$ and $f_1(b) = f_1(d)$. If $B_{ab} = \emptyset$ or $B_{cd} = \emptyset$ the theorem is proved. If $B_{ab} \neq \emptyset$, let $R \in B_{ab}$, then aRb. Also, $f_1(a)f_2(R)f_1(b)$ and $f_1(c)f_2(R)f_1(d)$. Since f is strong, cRd and $R \in B_{cd}$. Therefore $B_{ab} \subseteq B_{cd}$. Similarly, $B_{ab} \supseteq B_{cd}$ and $B_{ab} = B_{cd}$. Therefore f is a juncture homomorphism.

Theorem 11b. Let $f = \langle f_1, f_2 \rangle$ be a juncture homomorphism. Then $f^* = \langle f_1, f_3 \rangle$, where f_3 is the identity, is a bundle homomorphism.

Proof. Assume $f = \langle f_1, f_2 \rangle$ is a juncture homomorphism and there exist $a, b \in P$ such that $f_1(a) = f_1(b)$. Assume there exists $c \in P$ such that aMc. Then for every $R \in M$, aRc. f is full, hence $f_1(a)f_2(R)f_1(c)$. f is regular, hence there exists $d \in P$ such that $f_1(d) = f_1(c)$ and bRd. f is juncture, $f_1(a) = f_1(b)$ and $f_1(c) = f_1(d)$, and $B_{ac} \neq \emptyset$, hence $B_{bd} = B_{ac}$, hence bMd. Thus Condition (i) of the bundle equivalence and homomorphism is satisfied. Similarly for Condition (ii). Therefore, $f^* = \langle f_1, f_3 \rangle$ is a bundle homomorphism.   Q.E.D.

Theorem 12. Let $f: N \to N'$ be a juncture network homomorphism and C the multiplex graph derived from N. Then f induces a map from C to C' and C' is a multiplex graph.

Proof. Let $f: N \to N'$ be a juncture network homomorphism and let $C = \langle P, \mathfrak{M} \rangle$ be the multiplex graph derived from N. Let $M \in \mathfrak{M}$,

define $*f_2(M) = \{\langle f(a), f(b)\rangle : \langle a, b\rangle \in M\}$. Define $\mathfrak{M}' = \{*f_2(M): M \in \mathfrak{M}\}$, then $*f: C \to C' = \langle P', \mathfrak{M}'\rangle$ where $*f = \langle f_1, *f_2\rangle$ is a full network homomorphism. Suppose $*f_2(M_1) \cap *f_2(M_2) \neq \emptyset$ then there exists an $\langle f_1(a), f_1(b)\rangle \in *f_2(M_1) \cap *f_2(M_2)$ this implies there exist $x, y \in P$ and $c, d \in P$ such that $f_1(x) = f_1(a), f_1(y) = f_1(b), f_1(c) = f_1(a)$, $f_1(d) = f_1(b)$ and $\langle x, y\rangle \in M_1$ and $\langle c, d\rangle \in M_2$. This in turn implies that $B_{xy} = B_1$ where $M_1 = \{B: B = B_1\}$ and $B_{cd} = B_2$ where $M_2 = \{B: B = B_2\}$. Note that both $B_{xy} \neq \emptyset \neq B_{cd}$, $f_1(x) = f_1(c)$ and $f_1(y) = f_1(d)$. Because f is a juncture homomorphism $B_{xy} = B_{cd}$. But this implies that $B_1 = B_2$ and $M_1 = M_2$. Therefore $*f_2(M_1) \cap *f_2(M_2) \neq \emptyset$ implies $M_1 = M_2$ so C' is a multiplex graph. Q.E.D.

Theorem 13. Let $f: N \to N'$ be a juncture network homomorphism where $N = \langle P, \mathfrak{R}\rangle$, $N' = \langle P', \mathfrak{R}'\rangle$ and $f = \langle f_1, f_2\rangle$. If $\circ$ is relation composition and $\langle \mathfrak{R}, \circ\rangle$ is a semigroup then $f_2: \langle \mathfrak{R}, \circ\rangle \to \langle \mathfrak{R}', \circ\rangle$ is an isomorphism.

Proof. Let $\langle x, z\rangle \in f_2(R_1) \circ f_2(R_2)$, then there exist $y \in P'$ such that $\langle x, y\rangle \in f_2(R_1)$ and $\langle y, z\rangle \in f_2(R_2)$. Since f is regular there exists $a, b, c \in P$ such that $f(a) = x$, $f(b) = y$, and $f(c) = z$ such that $aR_1 b$ and $bR_2 c$. But this implies that $a(R_1 \circ R_2)c$ and $f_1(a)f_2(R_1 \circ R_2)f_1(c)$. So $\langle x, z\rangle \in f_2(R_1 \circ R_2)$ and $f_2(R_1) \circ f_2(R_2) \subseteq f_2(R_1 \circ R_2)$. Now suppose $\langle x, z\rangle \in f_2(R_1 \circ R_2)$. Then there exists $a, d \in P$ such that $f(a) = x$, $f(d) = z$ and $a(R_1 \circ R_2)d$. By definition there exists $c \in P$ such that $aR_1 c$ and $cR_2 d$ and $f_1(a)f_2(R_1)f_1(c)$ and $f_1(c)f_2(R_2)f(d)$ so $f_1(a)(f_2(R_1) \circ f_2(R_2))f_1(d)$ and $x(f_2(R_1) \circ f_2(R_2))z$ so $\langle x, z\rangle \in f_2(R_1) \circ f_2(R_2)$. Therefore $f_2(R_1) \circ f_2(R_2) = f_2(R_1 \circ R_2)$. So far in the proof we have used only regularity but we need yet to show that f_2 is one to one. Suppose $f_2(R_1) = f_2(R_2)$ where $R_1 \neq R_2$. Then there exist $\langle a, b\rangle$ in R_1 or R_2 but not the other. Suppose $\langle a, b\rangle \in R_1$ and $\langle a, b\rangle \notin R_2$, $\langle f_1(a), f_1(b)\rangle \in f_2(R_1)$ but since $f_2(R_1) = f_2(R_2)$, $\langle f_1(a), f_1(b)\rangle \in f_2(R_2)$. Since f is regular there exists $c \in P$ such that $f_1(a) = f_1(c)$ and $cR_2 b$. Therefore $B_{cb} \neq \emptyset$. Since $aR_1 b$, $B_{ab} \neq \emptyset$, $f_1(a) = f_1(c)$, and $f_1(b) = f_1(b)$ together with the definition of juncture homomorphism implies that $B_{ab} = B_{cb}$. But $cR_2 b$ so $R_2 \in B_{cb}$ and $R_2 \in B_{ab}$ and $aR_2 b$ which is a contradiction so f_2 is one to one and therefore an isomorphism. Q.E.D.

Theorem 14. If $N = \langle P, S\rangle$ and $N' = \langle P, S'\rangle$ are networks, $f_1: P \to P$ the identity, and $f_2: S \to S'$ a mapping then $f = \langle f_1, f_2\rangle$ is a weak full network homomorphism if and only if $f_2(R) = \cup[R]f_2$.

Proof.   Let $\langle x, y \rangle$ be in $f_2(R)$. Then by definition there is some Q in S and $\langle a, b \rangle$ in Q such that $f_2(Q) = f_2(R)$, $f_1(a) = x$, and $f_1(b) = y$. f_1 is the identity map so $a = x$ and $b = y$, thus $\langle x, y \rangle$ is in Q. Q is in $[R]f_2$ therefore $\langle x, y \rangle$ is in $\cup[R]f_2$. Conversely, if $\langle x, y \rangle$ is in $\cup[R]f_2$ there exists Q in S such that $f_2(Q) = f_2(R)$ and $\langle x, y \rangle$ is in Q. But since f_1 is the identity $\langle x, y \rangle = \langle f_1(x), f_1(y) \rangle$ is in $f_2(Q) = f_2(R)$. Therefore $f_2(R) = \cup[R]f_2$.

Conversely, if $\langle a, b \rangle \in f_2(R) = \cup[R]f_2$ there exists a relation Q in S such that $f_2(Q) = f_2(R)$ and $\langle a, b \rangle \in Q$. Therefore $\langle f_1, f_2 \rangle$ is a weak full network homomorphism.   Q.E.D.

Theorem 15.   Let $f = \langle f_1, f_2 \rangle : \langle P, S \rangle \to \langle P, S' \rangle$ be a weak full network homomorphism such that f_1 is the identity mapping. Then f_2 preserves composition if and only if $\cup[R_1]f_2 \circ \cup[R_2]f_2 = \cup[R_1 \circ R_2]f_2$.

Proof.   The proof follows directly from the definitions and Theorem 14.

Theorem 16.   The equivalence on S induced by a strong mapping is a strong equivalence and conversely every strong equivalence is induced by some strong mapping.

Proof.   The proof follows directly from the definitions.

Theorem 17.   Let $f = \langle f_1, f_2 \rangle : \langle P, S \rangle \to \langle P, S' \rangle$ be a weak full network homomorphism, f_1 the identity, f_2 a strong mapping, and $\langle S, \circ \rangle$ a semigroup. Then f_2 is a semigroup homomorphism if and only if $f_2(T) = f_2(R_1 \circ R_2)$ implies $T \subseteq R_1 \circ R_2$ for all T, R_1, R_2 in S.

Proof.   Assume f_2 preserves composition and is a strong mapping, and suppose $f_2(T) = f_2(R_1 \circ R_2)$. Let $(a, b) \in T$. Then $(a, b) \in f_2(T) = f_2(R_1 \circ R_2) = f_2(R_1) \circ f_2(R_2)$. Then there exists c such that $\langle a, c \rangle \in f_2(R_1)$ and $\langle c, b \rangle \in f_2(R_2)$. Therefore there exist Q_1 and Q_2 such that $f_2(Q_1) = f_2(R_1)$, $f_2(Q_2) = f_2(R_2)$, and $\langle a, c \rangle \in Q_1$ and $\langle c, b \rangle \in Q_2$. This implies that $\langle a, b \rangle \in Q_1 \circ Q_2$. But $Q_1 \circ Q_2 = Q_1 \circ R_2 = R_1 \circ R_2$ since f_2 is strong and $\langle a, b \rangle \in R_1 \circ R_2$. Therefore $T \subseteq R_1 \circ R_2$.

Conversely, we assume f_2 is strong and $f_2(T) = f_2(R_1 \circ R_2)$ implies $T \subseteq R_1 \circ R_2$ for each $T, R_1, R_2 \in S$. Let $(a, b) \in f_2(R_1 \circ R_2)$. Then there exists T such that $(a, b) \in T$ and $f_2(T) = f_2(R_1 \circ R_2)$. By assump-

tion $(a, b) \in R_1 \circ R_2 \subseteq f_2(R_1) \circ f_2(R_2)$. The reverse of this is obtained as in the first part of this proof, noting that $R_1 \circ R_2 \subseteq f_2(R_1 \circ R_2)$.

<div align="right">Q.E.D.</div>

Theorem 18. Let $f = \langle f_1, f_2 \rangle : \langle P, S \rangle \to \langle P, S' \rangle$ where f_1 is the identity map and f_2 is an inclusion mapping. Then f_2 preserves composition if and only if

(i) $f_2(R_1) = f_2(R_2)$ implies $f_2(T \circ R_1) = f_2(T \circ R_2)$ and $f_2(R_1 \circ T) = f_2(R_2 \circ T)$, and

(ii) $f_2(T) = f_2(R_1 \circ R_2)$ implies $T \subseteq \max[R_1]f_2 \circ \max[R_2]f_2$, for all T, R_1, R_2 in S.

Proof. Assume f_2 is an inclusion map which preserves composition. Suppose $f_2(R_1) = f_2(R_2)$, then $f_2(R_1 \circ T) = f_2(R_1) \circ f_2(T) = f_2(R_2) \circ f_2(T) = f_2(R_2 \circ T)$. Similarly $f_2(T \circ R_1) = f_2(T \circ R_2)$ for every T in S. Now suppose $f_2(T) = f_2(R_1 \circ R_2)$. Then $T \subseteq \max[T]f_2 = f_2(T) = f_2(R_1 \circ R_2) = f_2(R_1) \circ f_2(R_2) = \max[R_1]f_2 \circ \max[R_2]f_2$, so property (ii) is proven.

Conversely, assume f_2 is an inclusion mapping and satisfies conditions (i) and (ii). We know $f_2(R_1 \circ R_2) = \max[R_1 \circ R_2]f_2$, therefore if $\langle a, b \rangle \in f_2(R_1 \circ R_2)$ then there exists a T in S such that $f_2(T) = f_2(R_1 \circ R_2)$ and $\langle a, b \rangle$ is in T. By condition (ii) $T \subseteq \max[R_1]f_2 \circ \max[R_2]f_2 = f_2(R_1) \circ f_2(R_2)$ so $\langle a, b \rangle \in f_2(R_1) \circ f_2(R_2)$. Now if $\langle a, b \rangle \in f_2(R_1) \circ f_2(R_2)$ then there exists c such that $\langle a, c \rangle \in f_2(R_1)$ and $\langle c, b \rangle \in f_2(R_2)$. Therefore there exist Q_1 and Q_2 such that $f_2(Q_1) = f_2(R_1)$ and $f_2(Q_2) = f_2(R_2)$ and $\langle a, c \rangle \in Q_1$ and $\langle c, b \rangle \in Q_2$. Therefore $\langle a, b \rangle \in Q_1 \circ Q_2 \subseteq f_2(Q_1 \circ Q_2) = f_2(Q_1 \circ R_2) = f_2(R_1 \circ R_2)$ by condition (i). Therefore $f_2(R_1) \circ f_2(R_2) = f_2(R_1 \circ R_2)$.

<div align="right">Q.E.D.</div>

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