Ime in priimek

Vpisna številka

1	
2	
3	
Σ	

Linearna algebra: 2. kolokvij

29. maj 2024

Čas pisanja: 75 minut. Dovoljena je uporaba dveh listov velikosti A4 z obrazci. Uporaba elektronskih pripomočkov ni dovoljena. Rezultati bodo objavljeni na *ucilnica.fri.uni-lj.si*. **Vse odgovore dobro utemelji!**

1. naloga (30 točk)

Dana je matrika

$$A = \begin{array}{c} \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ -2 & 4 & 2 \end{bmatrix} \end{array}.$$

a) (15) Poišči ortonormirano bazo stolpčnega prostora $C(A)$.

$$\vec{u}_1 = \vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$1-2-8$

$$\|\vec{u}_1\| = \sqrt{1+4+4} = 3$$

$$\vec{u}_2 = \vec{a}_2 - \frac{\vec{a}_2\vec{u}_1}{\vec{u}_1\vec{u}_1}\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{-9}{9}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\|\vec{u}_2\| = \sqrt{4+1+4} = 3$$

$\vec{a}_3 = \vec{a}_1 + \vec{a}_2$, zato je $C(A) = \mathcal{L}in\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \mathcal{L}in\{\vec{a}_1, \vec{a}_2\}$ $\quad \vec{u}_3 = \vec{0}$

$$\vec{q}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$\{\vec{q}_1, \vec{q}_2\}$ je ONB za $C(A)$

b) (10) Poišči ortonormirano bazo za ortogonalni komplement $C(A)^\perp$ prostora $C(A)$.

$$C(A)^\perp = N(A^T)$$

$$A^T = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \\ 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 6 \\ 0 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \to x_1 = -2x_3 \\ \to x_2 = 2x_3 \\ x_3 \text{ param.} \end{array}$$

$$\vec{w} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\|\vec{w}\| = 3$$

$$\vec{q}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$\{\vec{q}_3\}$ je ONB za $C(A)^\perp$

c) (5) Vektor $\vec{v} = [1, 5, 1]$ zapiši kot vsoto $\vec{v} = \vec{v}_1 + \vec{v}_2$, kjer je $\vec{v}_1 \in C(A)$ in $\vec{v}_2 \in C(A)^\perp$.

$$\underline{\underline{\mathrm{proj}_{\vec{q}_3}(\vec{v})}} = (\vec{v} \cdot \vec{q}_3)\,\vec{q}_3 = \left(\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}\right)\vec{q}_3 = \left(-\frac{2}{3} + \frac{10}{3} + \frac{1}{3}\right)\vec{q}_3 = 3\vec{q}_3 = \underline{\underline{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \vec{v}_2}}$$

$$\vec{v}_1 = \vec{v} - \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}}} = \vec{v}_1$$

2. naloga (35 točk)

Pri nekih meritvah smo dobili vrednosti

x_i	0	1	2	3
y_i	−1	8	−9	10

Predpostavljamo, da odvisnost $y = f(x)$ opisuje funkcija oblike $f(x) = a \cdot (x-1)^2 + b \cdot 2^x + c$.

a) (15) Zapiši sistem enačb za neznane parametre a, b in c v obliki $A\vec{x} = \vec{f}$. Ali je dobljen sistem rešljiv?

$$\left.\begin{aligned} a + b + c &= -1 \\ 2b + c &= 8 \\ a + 4b + c &= -9 \\ 4a + 8b + c &= 10 \end{aligned}\right\} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 4 & 1 \\ 4 & 8 & 1 \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} -1 \\ 8 \\ -9 \\ 10 \end{bmatrix}$$

$$[A \mid \vec{f}\,] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & 8 \\ 1 & 4 & 1 & -9 \\ 4 & 8 & 1 & 10 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & 8 \\ 0 & 3 & 0 & -8 \\ 0 & 4 & 0 & 11 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \to 3b = -8 \to b = -\frac{8}{3} \\ \to 4b = 11 \to b = \frac{11}{4} \end{array} \Bigg\} \text{ sistem ni rešljiv}$$

b) (15) Izračunaj a, b in c v smislu linearne metode najmanjših kvadratov.

$$A^T A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 4 & 1 \\ 4 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 37 & 6 \\ 37 & 85 & 15 \\ 6 & 15 & 4 \end{bmatrix}$$

$$A^T \vec{f} = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 8 \\ -9 \\ 10 \end{bmatrix} = \begin{bmatrix} 30 \\ 59 \\ 8 \end{bmatrix}$$

$$[A^TA \mid A^T\vec{f}\,] = \left[\begin{array}{ccc|c} 18 & 37 & 6 & 30 \\ 37 & 85 & 15 & 59 \\ 6 & 15 & 4 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 18 & 37 & 6 & 30 \\ 0 & 161 & 48 & -48 \\ 0 & 4 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 18 & 37 & 6 & 30 \\ 0 & 161 & 48 & -48 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 18 & 37 & 0 & 36 \\ 0 & 161 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

$$\begin{array}{rcccc} 37 \cdot \mathrm{I} & 666 & 1369 & 222 & \mid 1110 \\ 18 \cdot \mathrm{II} & 666 & 1530 & 270 & \mid 1062 \\ \hline & 0 & 161 & 48 & -48 \end{array} \qquad \begin{array}{rcccc} 4\mathrm{II} & 0 & 644 & 192 & \mid -192 \\ 161\,\mathrm{III} & 0 & 644 & 483 & \mid -483 \\ \hline & 0 & 0 & 291 & \mid -291 \\ & 0 & 0 & 1 & \mid -1 \end{array}$$

$$\begin{array}{rcccc} \mathrm{I} & 18 & 37 & 6 & \mid 30 \\ 3\mathrm{III} & 18 & 45 & 12 & \mid 24 \\ \hline & 0 & 8 & 6 & \mid -6 \\ & 0 & 4 & 3 & \mid -3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 18 & 37 & 0 & 36 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 18 & 0 & 0 & 36 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right] \qquad \begin{aligned} a &= 2 \\ b &= 0 \\ c &= -1 \end{aligned}$$

c) (5) Izračunaj vrednost, ki jo za $x = 4$ napove dobljena funkcija f.

$$f(x) = 2(x-1)^2 + 0 \cdot 2^x - 1 = 2(x-1)^2 - 1$$

$$f(4) = 2 \cdot 3^2 - 1 = 18 - 1 = 17$$

2. naloga (35 točk)

Pri nekih meritvah smo dobili vrednosti

x_i	0	1	2	3
y_i	-1	8	-9	10

Predpostavljamo, da odvisnost $y = f(x)$ opisuje funkcija oblike $f(x) = a\cdot(x-1)^2 + b\cdot 2^x + c$.

a) (15) Zapiši sistem enačb za neznane parametre a, b in c v obliki $A\vec{x} = \vec{f}$. Ali je dobljen sistem rešljiv?

$$\left.\begin{aligned} a+b+c&=-1\\ 2b+c&=8\\ a+4b+c&=-9\\ 4a+8b+c&=10 \end{aligned}\right\} \quad A=\begin{bmatrix}1&1&1\\0&2&1\\1&4&1\\4&8&1\end{bmatrix} \quad \vec{f}=\begin{bmatrix}-1\\8\\-9\\10\end{bmatrix}$$

$$[A|\vec{f}\,] \sim \left[\begin{array}{ccc|c}1&1&1&-1\\0&2&1&8\\1&4&1&-9\\4&8&1&10\end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} a & b & c & \end{array}\\ \left[\begin{array}{ccc|c}1&1&1&-1\\0&2&1&8\\0&3&0&-8\\0&4&0&11\end{array}\right]\end{array} \begin{array}{l} \rightarrow 3b=-8 \rightarrow b=-\frac{8}{3}\\ \rightarrow 4b=11 \rightarrow b=\frac{11}{4}\end{array} \Bigg\}\ \text{sistem ni rešljiv}$$

b) (15) Izračunaj a, b in c v smislu linearne metode najmanjših kvadratov.

$$A^TA = \begin{bmatrix}1&0&1&4\\1&2&4&8\\1&1&1&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&2&1\\1&4&1\\4&8&1\end{bmatrix} = \begin{bmatrix}18&37&6\\37&85&15\\6&15&4\end{bmatrix}$$

$$A^T\vec{f} = \begin{bmatrix}1&0&1&4\\1&2&4&8\\1&1&1&1\end{bmatrix}\begin{bmatrix}-1\\8\\-9\\10\end{bmatrix} = \begin{bmatrix}30\\59\\8\end{bmatrix}$$

$$[A^TA|A^T\vec{f}\,] = \left[\begin{array}{ccc|c}18&37&6&30\\37&85&15&59\\6&15&4&8\end{array}\right] \sim \left[\begin{array}{ccc|c}18&0&6&30\\37&0&15&59\\6&0&4&8\end{array}\right] \sim \left[\begin{array}{ccc|c}3&0&1&5\\37&0&15&59\\3&0&2&4\end{array}\right] \sim \left[\begin{array}{ccc|c}3&0&1&5\\4&0&-7&15\\0&0&1&-1\end{array}\right] \sim$$

$b=0$

$$\begin{array}{rccc|c} 11\cdot\text{III} & 33 & 0 & 22 & 44\\ \text{II} & 37 & 0 & 15 & 59\\ \hline & 4 & 0 & -7 & 15 \end{array}$$

$$\left[\begin{array}{ccc|c}3&0&0&6\\4&0&0&8\\0&0&1&-1\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&2\\0&0&0&0\\0&0&1&-1\end{array}\right] \begin{array}{l}\rightarrow a=2\\ \\ \rightarrow c=-1\end{array}$$

c) (5) Izračunaj vrednost, ki jo za $x = 4$ napove dobljena funkcija f.

$$f(x) = 2(x-1)^2 + 0\cdot 2^x - 1 = 2(x-1)^2 - 1$$

$$f(4) = 2\cdot 3^2 - 1 = 18 - 1 = \underline{\underline{17}}$$

3. naloga (35 točk)

Naj bo

$$A = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

a) (10) Izračunaj karakteristični polinom matrike A in poišči vse lastne vrednosti matrike A.

$$\begin{vmatrix} 3-\lambda & 0 & -1 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ -1 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 \underbrace{\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix}}_{} = \underline{\underline{(\lambda-2)^3(\lambda-4)}}$$

$$(3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 9 - 1$$
$$= \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4)$$

$$\underline{\underline{\lambda_{1,2,3} = 2 \qquad \lambda_4 = 4}}$$

b) (10) Poišči vse (linearno neodvisne) lastne vektorje matrike A.

- $\lambda_{1,2,3} = 2$

$$A - 2I = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 = x_3$$

x_2, x_3, x_4 param.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- $\lambda_4 = 4$

$$A - 4I = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \begin{matrix} \rightarrow x_1 = -x_3 \\ \rightarrow x_2 = 0 \\ \\ \rightarrow x_4 = 0 \end{matrix}$$

x_3 param.

$$\vec{v}_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

c) (10) Ali obstajata ortogonalna matrika Q in diagonalna matrika D, da velja $A = QDQ^{\mathsf{T}}$? Če taki matriki Q in D obstajata, ju zapiši. Če ne, razloži, zakaj ne.

Obstajata, ker je A simetrična.

$2^7 = 128$
$4^7 = 16384$

$\vec{v}_i$ so paroma $\perp$, $\|\vec{v}_1\| = \|\vec{v}_3\| = 1$, $\|\vec{v}_2\| = \|\vec{v}_4\| = \sqrt{2}$

$$Q = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 4 \end{bmatrix}$$

d) (5) Izračunaj A^7.

$$A^7 = (QDQ^{\mathsf{T}})^7 = QD^7Q^{\mathsf{T}} = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2^7 & & & \\ & 2^7 & & \\ & & 2^7 & \\ & & & 4^7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2^7 & 0 & 0 \\ 2^7/\sqrt{2} & 0 & 2^7/\sqrt{2} & 0 \\ 0 & 0 & 0 & 2^7 \\ -4^7/\sqrt{2} & 0 & 4^7/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2^6+2^{13} & 0 & 2^6-2^{13} & 0 \\ 0 & 2^7 & 0 & 0 \\ 2^6-2^{13} & 0 & 2^6+2^{13} & 0 \\ 0 & 0 & 0 & 2^7 \end{bmatrix}$$