

Linear least squares method. Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix with $m \geq n$. Let $\mathbf{b} \in \mathbb{R}^m$ be a vector. How would you find the orthogonal projection of \mathbf{b} on to the column space of A , $C(A)$? (Assume that the columns of A are linearly independent.)

1. We want to approximate a real function f on the interval $[a, b]$ with a polynomial. We will do this (perhaps naïvely) by dividing the interval $[a, b]$ with $k + 1$ equidistant points $a = x_0, x_1, \dots, x_k = b$ and then find the coefficients of the polynomial $p(x)$ that is the best fit to the data in the table below according to the linear least squares method.

| | | | | | |
|----------|----------|---------|----------|---------|----------|
| x_0 | x_1 | \dots | x_i | \dots | x_k |
| $f(x_0)$ | $f(x_1)$ | \dots | $f(x_i)$ | \dots | $f(x_k)$ |

- (a) Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial of degree n . Write the matrix A of the corresponding linear system and the right-hand side of \mathbf{b} according to the data in the table above.
 - (b) Find the approximations of orders 0, 1 and 2 for the function $f(x) = \frac{x^2}{1+x^2}$ on the interval $[-1, 1]$ using the points $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.
 - (c) Using octave approximate the function $g(x) = \frac{1}{1+25x^2}$ on $[-1, 1]$ with polynomials of order 0, 2, ..., 20, dividing the interval $[-1, 1]$ with 21 equidistant points. Find the approximations for the exact data and for data with (artificially added) errors. Using the `plot` command plot the graphs of the original functions and all the approximations.
2. Use the linear least squares method to solve the following problem: In the plane \mathbb{R}^2 we have n transmitters in known locations $(p_1, q_1), \dots, (p_n, q_n)$. A receiver can measure the distances from all the transmitters d_1, \dots, d_n . We want to determine the position of the receiver given the distances from the transmitters. In the ideal case the measurements are perfect and for each $i = 1, \dots, n$ we have an equation

$$(x - p_i)^2 + (y - q_i)^2 = d_i^2.$$

The solution of this system of equations then determines the unknown position (x, y) .

- (a) The first problem is that the equations are *not* linear. But the difference of two consecutive equations is linear. Write down these differences to obtain a system of $n - 1$ linear equations.
- (b) Write the matrix $A \in \mathbb{R}^{(n-1) \times 2}$ of the system and the corresponding right-hand side $\mathbf{b} \in \mathbb{R}^{n-1}$. Additional problem is that the measurements are not exact which means that the system $A\mathbf{x} = \mathbf{b}$ almost surely has no solution.
- (c) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$. Write the octave function `X = sprejemnik([pi, qi], [di])` that finds the position of the receiver $X(x, y)$ given the transmitter positions (p_i, q_i) and distances d_i . (The positions (p_i, q_i) are contained in an $n \times 2$ matrix and the distances d_i are given in a column matrix of length n . The result X should be row vector $X = [x, y]$.)

- (d) Test the function using artificial data from the files `oddajniki.txt` and `razdalje.txt` found on *ucilnica*. You can import them into octave using the `load` command. Display the results.