Mathematical modelling
3.2 Parametric curves

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2019/2020
3.2. Parametric curves

A parametric curve (or parametrized curve) in $\mathbb{R}^m$ is a vector function $f : \mathbb{R} \to \mathbb{R}^m$, or from $I \to \mathbb{R}^m$ where $I \subset \mathbb{R}$ is a bounded or unbounded interval

$$f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_m(t) \end{bmatrix},$$

The independent variable (in this case $t$) is the parameter of the curve.

For every value $t \in I$, $f(t)$ represents a point in $\mathbb{R}^m$.

As $t$ runs through $I$, $f(t)$ traces a path, or a curve, in $\mathbb{R}^m$. 
If $m = 2$, then for every $t \in I$,

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = r(t)$$

is the position vector of a point in the plane $\mathbb{R}^2$.

All points $\{f(t), t \in I\}$ form a plane curve:

In this example $x(t) = t \cos t, y(t) = t \sin t, t \in [-3\pi/4, 3\pi/4]$
If \( m = 3 \), then
\[
    f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \mathbf{r}(t)
\]
is the position vector of a point in \( \mathbb{R}^3 \) for every \( t \), and \( \{ f(t), \ t \in I \} \) is a space curve:

In this example \( x(t) = \cos t, \ y(t) = \sin t, \ z(t) = t/5, \ t \in [0, 4\pi] \)
Examples:

\[ f(t) = \begin{bmatrix} 2 \cos t \\ 2 \sin t \end{bmatrix}, \quad t \in [0, 2\pi] \]

a circle with radius 2 and center (0, 0)

\[ f(t) = r_0 + te, \quad t \in \mathbb{R}, \]
\[ r_0, e \in \mathbb{R}^m, \quad e \neq 0 \]

line through \( r_0 \) in the direction of \( e \) in \( \mathbb{R}^m \)

m=2:
- slope \( k = e_2/e_1 \) if \( e_1 \neq 0 \)
- vertical if \( e = (0, e_2) \)
- horizontal if \( e = (e_1, 0) \)
A parametric curve $f(t), t \in [a, b]$ is closed if $f(a) = f(b)$.

Example:

$$f(t) = \begin{bmatrix} \cos 3t \\ \sin 5t \end{bmatrix}, t \in [0, 2\pi]$$

**Lissajous curves:** $x(t) = \sin(nt + \delta), y(t) = \sin mt$, are closed if the ratio $n/m$ is rational. They describe 2D harmonic motion.
Problem: What path does the valve on your bicycle wheel trace as you bike along a straight road?

Represent the wheel as a circle of radius $a$ rolling along the $x$-axis, the valve as a fixed point on the circle, the parameter is the angle of rotation:

The curve is a cycloid: $x(\theta) = a\theta - a\sin \theta$, $y(\theta) = a - a\cos \theta$. 
A parametric curve \( f(t) \) describes the motion of a point with respect to \( t \). The path that it traces is simply a *curve* \( C \).

The following parametric curves all describe the circle with radius \( a \) around the origin (as well as many others):

\[
f_1(t) = \begin{bmatrix} a \sin t \\ a \cos t \end{bmatrix}, \quad t \in [0, 2\pi]
\]

\[
f_2(t) = \begin{bmatrix} a \cos 2t \\ a \sin 2t \end{bmatrix}, \quad t \in [0, 2\pi]
\]

\[
f_3(t) = \begin{bmatrix} a \cos t \\ a \sin t \end{bmatrix}, \quad t \in \mathbb{R}
\]
Problem: find the self-intersection (if there is one) of a parametric curve

Let \( f(t) = \begin{bmatrix} t^3 - 2t \\ t^2 - t \end{bmatrix} \)

A self-intersection is at a point \( f(t) = f(s) \), with \( t \neq s \), so:

\[
\begin{align*}
    t^3 - 2t &= s^3 - 2s, \\
    t^2 - t &= s^2 - s, \\
    t^3 - s^3 &= 2t - 2s, \\
    t^2 - s^2 &= t - s
\end{align*}
\]

Since \( t \neq s \) we can divide by \( t - s \):

\[
\begin{align*}
    t^2 + ts + s^2 &= 2, \\
    t + s &= 1, \\
    t &= 1 - s, \\
    (1 - s)^2 + s(1 - s) + s^2 &= 2.
\end{align*}
\]

The self-intersection (where \( s \) and \( t \) can be interchanged) is at

\[
    s = \frac{1 + \sqrt{5}}{2}, \quad t = \frac{1 - \sqrt{5}}{2}, \quad f(t) = f(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
Problem: do two parametric curves intersect. Imagine two superheroes speeding along the two curves. Do they meet?

Let \( f_1(t) = \begin{bmatrix} t^2 - 1 \\ -t^3 - t^2 + t + 1 \end{bmatrix}, \quad f_2(s) = \begin{bmatrix} s - 1 \\ 1 - s^2 \end{bmatrix}. \)

To find the intersections, solve the system

\[
\begin{align*}
  t^2 - 1 &= s - 1, \\
  -t^3 - t^2 + t + 1 &= 1 - s^2 \\
  s &= t^2 \\
  -s^6 - s^4 + s^2 + 1 &= 1 - s^2
\end{align*}
\]

There are three solutions (work them out!!):

\[
\begin{align*}
  t &= -1, s = 1 & x &= 0, y &= 0 \\
  t &= 0, s = 0 & x &= -1, y &= 1 \\
  t &= 1, s = 1 & x &= 0, y &= 0
\end{align*}
\]

The superheroes meet at \( t = 0, s = 0 \) at the point \((-1, 1)\) and at \( t = 1, s = 1 \) at the point \((0, 0)\).
The derivative of the vector function \( f(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} \) at the point \( a \) is the vector:

\[
Df(a) = \begin{bmatrix} x'_1(a) \\ \vdots \\ x'_m(a) \end{bmatrix} = f'(a) = \lim_{h \to 0} \frac{1}{h}(f(a + h) - f(a))
\]

The vector \( f'(a) \) (if it exists) represents the velocity vector of a point moving along the curve at the point \( t = a \).

If \( f'(a) \neq 0 \) it points in the direction of the tangent at \( t = a \).
The **linear approximation** of $f$ at $t = a$ is

$$L_a(t) = f(a) + (t - a)f'(a)$$

If $f'(a) \neq 0$, this is a parametric line corresponding to the tangent line to the curve $f(t)$ at $t = a$.

In this case $f(a)$ is a **regular point** of the parametric curve and the parametric curve is **smooth** at $t = a$.

If $f'(a) = 0$ (or if it does not exist), the point $f(a)$ is **singular**.
A curve \( C \in \mathbb{R}^m \) is *smooth* at a point \( x \) on \( C \) if there exists a parametrization \( f(t) \) of \( C \), such that \( f(a) = x \) and \( f'(a) \neq 0 \).

A smooth curve has a tangent at every point \( x \in C \).

**Problem:** Is the curve \( C = \{ f(t), t \in [0, \sqrt{2\pi}] \} \),
\[
f(t) = \begin{bmatrix} \cos(t^2) \\ \sin(t^2) \end{bmatrix},
\] smooth?

Since \( x^2 + y^2 = 1 \), \( f(t) \) is a parametrization of the unit circle which is a smooth curve (it has a tangent at every point).

Since \( f'(0) = 0 \) the parametrization \( f(t) \) is not a smooth at \( t = 0 \).

Find a smooth parametrization!
Problem: is the cycloid a smooth curve?

Our parametrization

\[ f(t) = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix} , \quad f'(t) = \begin{bmatrix} 1 - \cos t \\ \sin t \end{bmatrix} . \]

is not smooth at \( t = 2k\pi \) since \( f'(2k\pi) = 0 \).

Does a tangent exist? (It seems not, but let’s check...)
The slope of the tangent line at a point \( f(t) \) is:

\[
k_t = \frac{y'(t)}{x'(t)} = \frac{a \sin t}{a(1 - \cos t)}
\]

The left and right limits as \( t \to 2k\pi \) are

\[
\lim_{t \to 2k\pi^+} k_t = \lim_{t \to 2k\pi^+} \frac{\cos t}{\sin t} = -\infty, \quad \lim_{t \to 2k\pi^-} k_t = \lim_{t \to 2k\pi^-} \frac{\cos t}{\sin t} = \infty,
\]

so at these points the curve forms a sharp spike (a \textit{cusp}) and a tangent does not exist.

So, the cycloid is not smooth at the points where it touches the \( x \) axis.

(l'Hospital’s rule was used to compute the limits.)
**How long is the path of a point moving along a parametric curve?**

For example, how what distance does a point on the circle cover when the circle makes one full turn?

The *arc length* $s$ of a parametric curve $f(t)$, $t \in [a, b]$, in $\mathbb{R}^m$ is the length of the curve between the points $t = a$ in $t = b$. 
An approximate value for $s$ is the length of a polygonal curve connecting close enough points on curve:

$$s_n = \sum_{i=1}^{n} \| f(t_i) - f(t_{i-1}) \|$$

For $n$ big enough, $s_n$ is a practical approximation for $s$. 
If the function $f(t)$ is continuously differentiable, then we can approximate the value $f(t_i) = f(t_{i-1} + \Delta t)$, where $\Delta t = t_i - t_{i-1}$, by the linear approximation:

$$f(t_i) = f(t_{i-1}) + f'(t_{i-1}) \Delta t$$

and we get:

$$\|f(t_i) - f(t_{i-1})\| \approx \|f'(t_{i-1})\| \Delta t$$

$$s_n \approx \sum_{i=1}^{n} \|f'(t_{i-1})\| \Delta t.$$  

This is a Riemann integral sum of the function $\|f'(t)\|$.

In the limit as $n \to \infty$, $s_n$ converges to $s$ and the integral sum to the integral, so

$$s = \lim_{n \to \infty} s_n = \int_{a}^{b} \|f'(t)\| \, dt$$
Problem: The length of the path traced by a point on the circle after a full turn,

that is, of a cycle of the cycloid \( f(t) = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix} \):

\[
\begin{align*}
  s &= \int_{0}^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} \, dt \\
  &= \int_{0}^{2\pi} \sqrt{2 - 2\cos t} \, dt \\
  &= \int_{0}^{2\pi} \sqrt{4\sin^2 (t/2)} \, dt \\
  &= \int_{0}^{2\pi} 2\sin (t/2) \, dt \\
  &= -4(\cos(\pi) - \cos(0)) = 8
\end{align*}
\]
Problem: The arc length of the helix \( f(t) = \begin{bmatrix} a \cos t \\ a \sin t \\ bt \end{bmatrix} \), \( 0 \leq t \leq 2\pi \),

... is homework:)

Problem: The circumference of the ellipse \( \begin{bmatrix} a \cos t \\ b \sin t \end{bmatrix} \), \( a \neq b \)

\[
\int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, dt = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t} \, dt = 4a E(e)
\]

where \( e = \sqrt{1 - (b/a)^2} \) is its eccentricity and the function \( E \) is the nonelementary elliptic integral of 2nd kind.
Arc length from the initial $t = a$ to an arbitrary $t$

$$s(t) = \int_a^t \|f'(u)\| \, du$$

is an increasing function of $t$, so it has an inverse $t(s)$

So, the original parameter $t$ can be expressed as a function of the arc length $s$.

Inserting this into the parametrization gives the same curve with a different parametrization: $g(s) = f(t(s))$.

Arc length is called the *natural parameter* of the curve.
A curve $C$ is parametrized with the natural parameter $s$ has $\|f'(s)\| = 1$

A parametrization with the natural parameter is the *unit speed parametrization*.

The natural parametrization of a curve is extremely important in theory, but for practical computing it is less useful.
Example:

The standard parametrization of the circle

\[ f(t) = \begin{bmatrix} a \cos t \\ a \sin t \end{bmatrix} \]

is not the natural parametrization if \( a \neq 1 \), since

\[ \| f'(t) \| = \sqrt{a^2 \cos^2 t + a^2 \sin^2 t} = a \neq 1. \]

Since \( s(t) = \int_0^t a \, dt = at \) it follows that \( t = s/a \) and the natural parametrization is

\[ g(s) = \begin{bmatrix} a \cos(s/a) \\ a \sin(s/a) \end{bmatrix}. \]