Mathematical modelling
Chapter 3
Dynamical modelling: differential equations

Fakulteta za računalništvo in informatiko
Univerza v Ljubljani

2019/2020
4. Differential equations and dynamic models

Ordinary differential equation, ODE, is an equation of an unknown function and an independent variable. ODE relates the independent variable with the function and its derivatives.

If \( t \) is an independent variable, \( x(t) \) is a function of \( t \), then the ODE is of the form:

\[
F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0.
\]

Similarly if \( x \) is an independent variable, \( y(x) \) a function of \( x \), then the ODE is of the form:

\[
F(x, y, y', y'', \ldots, y^{(n)}) = 0.
\]

The order of a differential equation is the order of the highest derivative.
Examples of ODE’s

- \( \dot{x} - 3t^2 = 0. \)

  So,
  
  \[
  \frac{dx}{dt} = 3t^2 \implies x(t) = t^3 + C, \quad \text{where } C \text{ is a constant.}
  \]

  If we want to determine \( C \), we need an additional condition, e.g., *initial condition* \( x(0) = x_0, \) \( x_0 \in \mathbb{R}, \) or any other condition \( x(t_0) = x_0, \) \( x_0 \in \mathbb{R}. \)

- \( y''(x) + 2y'(x) = 3y(x). \)

  We will learn how to solve such an ODE, but right now let us only check that \( y(x) = Ce^{-3x}, \) \( C \in \mathbb{R} \) a constant, is a solution:

  - Calculate \( y''(x), y'(x): \)

    \[
    y'(x) = -3Ce^{-3x}, \quad y''(x) = 9Ce^{-3x}.
    \]

  - Plug into the given ODE:
    
    \[
    9Ce^{-3x} - 6Ce^{-3x} = 3Ce^{-3x}.
    \]
Another example of an ODE and a definition of a partial differential equation

\[ \cos t \cdot \ddot{x} - 3t^4 \cdot \dot{x} + 5e^t = 0. \]

Such ODE's cannot be solved analytically (or are at least hard to solve). We will learn how to solve such ODE's by using numerical methods.

**Partial differential equation, PDE,** is an equation for an unknown function \( u \) of \( n \geq 2 \) independent variables, e.g., for \( n = 2 \) we have

\[ F(x, y, u_x, u_y, u_{xx}, \ldots) = 0, \]

where \( x, y \) are the independent variables.

We will not consider PDE's, from now on DE means an ODE.
Applications of DEs

Differential equations are used for modelling a deterministic process: a law relating a certain quantity depending on some independent variable (for example time) with its rate of change, and higher derivatives.

1. Newton’s law of cooling:

\[ \dot{T} = k(T - T_\infty), \quad (1) \]

where \( T(t) \) is the temperature of a homogeneous body (can of beer) at time \( t \), \( T_0 \) is the initial temperature at time \( t_0 = 0 \), \( T_\infty \) is the temperature of the environment, \( k \) is a constant (heat transfer coefficient).

(1) is an example of a separable ODE and also the first order linear ODE. We will see shortly how to solve such types of ODE’s. For now you can check easily by yourself that the solution is

\[ T(t) = (T_0 - T_\infty)e^{kt}. \]
2. **Radioactive decay:**

\[ \dot{y}(t) = -ky(t), \quad k = \frac{\log 2}{t_{1/2}}, \]

where \( y(t) \) is the remaining quantity of a radioactive isotope at time \( t \), \( t_{1/2} \) is the *half-life* and \( k \) is the *decay constant*. The solution is

\[ y(t) = Ce^{-kt}, \quad \text{where } C \text{ is a constant.} \]

Let’s verify, that \( t_{1/2} \) really represents the time in which the amount of the isotope decreases to half of its current amount. At time \( t = 0 \) the amount is \( y(0) = Ce^0 = C \). We have to check that \( y(t_{1/2}) = \frac{C}{2} \):

\[ y(t_{1/2}) = Ce^{-k \log 2} = Ce^{-\log 2} = Ce^{\log 1/2} = \frac{C}{2}. \]

3. **Simple harmonic oscillator:**

\[ \ddot{x} + \omega x = 0. \]
The function \( x(t) \) is a \textit{solution} of a DE

\[ F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0 \]

on an interval \( I \) if it is at least \( n \) times differentiable and satisfies the identity

\[ F(t, x(t), \dot{x}(t), \ddot{x}(t), \ldots, x^{(n)}(t)) = 0 \]

for all \( t \in I \).

Analytically solving a differential equation is typically very difficult, very often impossible.

To find approximate solutions we use different simplifications and numerical methods.
4.1. First order ODE’s

Definition of a first order DE

We will (mostly) consider first order ODE’s in the form

\[ \dot{x} = f(t, x). \]

- The **general solution** is a one-parametric family of solutions
  \[ x = x(t, C). \]
- A **particular solution** is a specific function from the general solution, that usually satisfies some initial condition \( x(t_0) = x_0. \)
- A **singular solution** is an exceptional solution that is not part of the general solution.

We will first look at some simple types of 1.-st order DE’s that are analytically solvable.
A *separable* DE is of the form

\[ \dot{x} = f(t)g(x). \]  

(2)

This can be solved by:

- Inserting \( \dot{x} = \frac{dx}{dt} \) into (2):

\[ \frac{dx}{dt} = f(t)g(x). \]  

(3)

- *Separating variables* in (3):

\[ \frac{dx}{g(x)} = f(t) \, dt. \]  

(4)

- Integrating both sides of (3):

\[
\int \frac{1}{g(x)} \, dx = \int f(t) \, dt + C
\]
Example 1 of a separable DE

\[ \dot{x} = kx \]  where \( k \in \mathbb{R} \) is a fixed real number \hspace{1cm} (5)

\[ \frac{dx}{dt} = kx, \]

\[ \frac{dx}{x} = kdt, \]

\[ \log |x| = \int \frac{dx}{x} = \int k \, dt = kt + C, \]

where \( C \) is a constant and so

\[ |x| = e^{kt+C} \]

is a general solution to (5). Clearly, \( x(t) = 0 \) is also a solution of the equation. By introducing a new constant \( e^C \) which, by abuse of notation, we again denote by \( C \), this is equivalent to

\[ x(t) = Ce^{kt}, \, C \in \mathbb{R}. \]
Example 2 of a separable DE

\[
\dot{x} = kx(1 - x) \quad \text{where } k \in \mathbb{R} \text{ is a fixed real number} \quad (6)
\]

\[
\frac{dx}{dt} = kx(1 - x),
\]

\[
\frac{dx}{x(1 - x)} = kdt,
\]

By the method of partial fraction we get

\[
\log \left| \frac{x}{1 - x} \right| = \log |x| - \log |1 - x| = \int \frac{dx}{x} - \int \frac{dx}{1 - x} = \int k \, dt = kt + C,
\]

where \(C\) is a constant and so

\[
\frac{x}{1 - x} = Ce^{kt}.
\]

Expressing \(x(t)\) we get

\[
x(t) = \frac{1}{Ce^{-kt} + 1} \quad (7)
\]

is a general solution to (6). \(x(t)\) from (7) is called a logistic function.
Example 3 of a separable DE

\[ y' = \frac{-x}{ye^{x^2}}, \quad y(0) = 1. \]  \hspace{1cm} (8)

\[ \frac{dy}{dx} = \frac{-x}{ye^{x^2}}, \]

\[ ydy = -xe^{-x^2} \, dx, \]

Integrating:

\[ \frac{y^2}{2} = \int ydy = \int (-xe^{-x^2}) \, dx = \frac{1}{2}e^{-x^2} + C, \]

where \( C \) is a constant.

\[ \frac{1}{2} = \frac{y^2(0)}{2} = \frac{1}{2} + C \implies C = 0. \]

Expressing \( y(x) \) we get \( y(x) = \pm \sqrt{e^{-x^2}} \) and since \( y(0) > 0 \) we have

\[ y(x) = \sqrt{e^{-x^2}}. \]
Let $x(t)$ be the size of a population (bacteria, trees, people, . . .) at time $t$. The most common models for population growth are:

- **exponential growth**: the growth rate is proportional to the size, modelled by $\dot{x} = kx$, with the solution the exponential function $x(t) = x_0 e^{kt}$, where $x_0 = x(0)$ is the initial population size.

- **logistic growth**: the growth rate is proportional to the size and the resources, modelled by $\dot{x} = kx(1 - x/x_{max})$, where $x_{max}$ is the capacity of the environment, i.e., maximal population size that it still supports, with the solution is the logistic function.

- **general model**: the growth rate is proportional to the size, but the proportionality factor depends on time and size, modelled by $\dot{x} = k(x, t)f(x)$; the equation is not separable and is analytically solvable only in very specific cases.
$x(t)$ is the ratio of people in a given group that at time $t$ knows a certain piece of information.

Let $x_0 = x(t_0)$ be the 'informed' ratio at time $t = t_0$.

Consider two possible models:

- spreading through an external source: the rate of change is proportional to the uninformed ratio $\dot{x} = k(1 - x)$ with $x_0 = 0$,
- spreading through "word of mouth" the rate of change is proportional to the number of encounters between informed and uninformed members $\dot{x} = kx(1 - x)$ logistic law, again, with $x_0 > 0$. 

Real life example 2 - information spreading
First order linear DE

A \textit{first order linear DE} is of the form

\[ \dot{x} + f(t)x = g(t) \quad (9) \]

The equation is \textit{homogeneous} if \( g(t) = 0 \) and \textit{nonhomogenous} if \( g(t) \neq 0 \).

A homogeneous part of (9),

\[ \dot{x} + f(t)x = 0 \quad (10) \]

has a general solution of the form

\[ Cx_h(t), \quad (11) \]

where \( C \in \mathbb{R} \) is a constant and \( x_h(t) \) is a particular solution. Indeed:

\[ \text{Every } x(t) \text{ of the form (11) is a solution of (10):} \]

\[ x'(t) + f(t)x(t) = (Cx_h)'(t) + f(t)Cx_h(t) \]

\[ = Cx'_h(t) + f(t)Cx_h(t) \]

\[ = C(x'_h(t) + f(t)x_h(t)) \]

\[ = 0 \]
If \( x(t) \) is a solution of (10), then it must be of the form (11). Indeed, since \( x(t) \) and \( x_h(t) \) both solve (10),

\[
\left( \frac{x(t)}{x_h(t)} \right)' = \frac{x'(t)x_h(t) - x(t)x'_h(t)}{x_h^2(t)} = \frac{-f(t)x(t)x_h(t) + f(t)x(t)x_h(t)}{x_h^2(t)} = 0.
\]

Hence, \( \frac{x(t)}{x_h(t)} = C \) for some constant \( C \) and \( x(t) \) is of the form (11).

Let \( x_p(t) \) be any particular solution of (9):

\[
g(t) = x'_p(t) + f(t)x_p(t) \quad (12)
\]

The general solution of (9) is a sum

\[
x(t) = Cx_h(t) + x_p(t). \quad (13)
\]
First order linear DE

Indeed:

▶ Every \( x(t) \) of the form (13) is a solution of (9):

\[
x'(t) + f(t)x(t) = (Cx_h(t) + x_p(t))' + f(t)(Cx_h(t) + x_p(t)) \\
= Cx'_h(t) + x'_p(t) + f(t)Cx_h(t) + f(t)x_p(t) \\
= (Cx'_h(t) + f(t)Cx_h(t)) + (x'_p(t) + f(t)x_p(t)) \\
= 0 + g(t),
\]

where we used (12) in the last equality.

▶ If \( x(t) \) is a solution of (9), then it must be of the form (13). Indeed, since \( x(t) \) and \( x_p(t) \) both solve (9), \( x(t) - x_p(t) \) solves the homogenous part (10) of (9). Hence, \( x(t) - x_p(t) = Cx_h(t) \) for some \( C \) and \( x(t) = Cx_h(t) + x_p(t) \).

The particular solution \( x_p \) can be obtained by \textit{variation of the constant}, that is, by substituting the constant \( C \) is the homogenous solution by an unknown function \( C(t) \) which is then determined from the equation.
Example of a linear DE

\[ t^2 \dot{x} + tx = 1, \quad x(1) = 2. \quad \tag{14} \]

1. The homogenous part is

\[ t^2 \dot{x} + tx = 0. \quad \tag{15} \]

So the solution \( x_h \) to (15) is

\[
\begin{align*}
t^2 \, dx &= -tx \, dt \\
\Rightarrow \quad \frac{dx}{x} &= -\frac{dt}{t} \\
\Rightarrow \quad \log |x| &= -\log |t| + \log C = \log \frac{C}{|t|} \\
\Rightarrow \quad x_h &= \frac{C}{t}.
\end{align*}
\]

2. A particular solution of the nonhomogenous equation is obtained by variation of the constant:

\[
x = \frac{C(t)}{t}, \quad \dot{x} = \frac{C'(t)t - C(t)}{t^2}
\]

by inserting into (14) we obtain

\[
C'(t)t - C(t) + C(t) = 1 \quad \Rightarrow \quad C'(t) = \frac{1}{t} \quad \Rightarrow \quad C(t) = \log |t|.
\]
3. So the general solution of the nonhomogenous equation is

\[ x(t) = \frac{C}{t} + \frac{\log |t|}{t}. \quad (16) \]

4. Finally, since \( x(1) = 2 \), we get by plugging \( t = 1 \) into (16)

\[ 2 = x(1) = C \]

and hence the solution of (14) is

\[ x(t) = \frac{2 + \log |t|}{t}. \]
General solution of a linear DE

\[ y'(x) = f(x)y(x) + g(x). \]  \hfill (17)

1. The homogenous part is

\[ y'(x) = f(x)y(x). \]  \hfill (18)

So the solution \( y(x) \) to (18) is

\[
\log |y| = \int \frac{dy}{y} = \int f(x) \, dx + C \implies y(x) = C \cdot e^{\int f(x) \, dx}
\]

2. A particular solution of the nonhomogenous equation is obtained by the variation of the constant:

\[
y(x) = C(x) \cdot e^{\int f(x) \, dx}
\]

\[
y'(x) = C'(x) \cdot e^{\int f(x) \, dx} + C(x)f(x)e^{\int f(x) \, dx}
\]

By inserting into (17) we obtain

\[
C'(x) \cdot e^{\int f(x) \, dx} + C(x)f(x)e^{\int f(x) \, dx} = f(x)C(x) \cdot e^{\int f(x) \, dx} + g(x)
\]
General solution of a linear DE

Hence

\[ C'(x) \cdot e^{\int f(x)dx} = g(x), \]

and so

\[ C(x) = \int (g(x)e^{-\int f(x)dx})dx. \]

Finally the solution is

\[ y(x) = e^{\int f(x)dx} \left( C + \int (g(x)e^{-\int f(x)dx})dx \right). \]

In the example \( t^2 \dot{x} + tx = 1 \) (or \( \dot{x} = -\frac{1}{t}x + \frac{1}{t^2} \)) above we get

\[ x(t) = e^{\int -\frac{1}{t}dt} \left( C + \int \left( \frac{1}{t^2} e^{\int \frac{1}{t}dt} \right)dt \right) \]

\[ = e^{\log |\frac{1}{t}|} \left( C + \int \left( \frac{1}{t^2} t \right)dt \right) \]

\[ = \frac{1}{t} \left( C + \log |t| \right). \]
Real life example - Newton’s second law

A ball of mass $m$ kg is thrown vertically into the air with initial velocity $v_0 = 10$ m/s. We follow its trajectory. By Newton’s second law of motion,

$$F = ma,$$

where $m$ is the mass, $a = \dot{v} = \ddot{x}$ is acceleration and $v$ velocity, and $F$ is the sum of forces acting on the ball.

- Assuming no air friction the model is

  $$m \dot{v} = -mg,$$
  
  where $g$ is the gravitational constant. The solution is
  
  $$v = -gt + C$$
  
  where $C$ is a constant.

- Assuming the linear law of resistance (drag) $F_u = -kv$ the model is

  $$m \dot{v} = -mg - kv.$$

The solution is $v = v_h + v_p$ where

$$v_h = Ce^{-kt/m}$$

and

$$v_p = -mg/k.$$
Motion of ball in the case $m = 1$, $k = 1$ and approximating $g \approx 10$ (we will omit units)

<table>
<thead>
<tr>
<th>Model</th>
<th>Velocity and position</th>
<th>Solution</th>
</tr>
</thead>
</table>

$ma = -mg$
$\dot{v} = -10$

$v(t) = -10t + 10$
$x(t) = -5t^2 + 10t$

$ma = -mg - kv$
$\dot{v} = -v - 10$

$v(t) = 20e^{-t} - 10$
$x(t) = 20 - 20e^{-t} - 10t$
The ball reaches the top at time $t$ where $v(t) = 0$ and the ground at time $t$ where $x(t) = 0$.

- Assuming no friction, the ball is at the top at $t = 10$.

At time $t = 1$, $x(t) = 0$, so it takes the same time going up and falling down.

- Assuming linear friction, the ball reaches the top at $t = \log 2$.

At time $2 \log 2$, $x(2 \log 2) = 20 - 5 - 20 \log 2 > 0$ so it takes longer falling down than going up.