1. A curve given in polar coordinates is easily translated into parametrization:

$$x(t) = r(t)\cos(t),$$
$$y(t) = r(t)\sin(t).$$

In our case $r(t) = a\sqrt{\cos(2t)}$, therefore:

$$x(t) = a\cos(t)\sqrt{\cos(2t)},$$
$$y(t) = a\sin(t)\sqrt{\cos(2t)}.$$

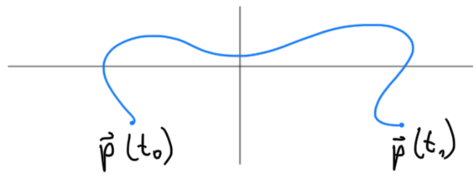
To evaluate the area we can use the description of the lemniscate in polar coordinates:

$$A = \frac{1}{2}\int_{\varphi_0}^{\varphi_1} r^2(\varphi)\,d\varphi = \frac{1}{2}\int_{-\pi/4}^{\pi/4} a^2\cos(2\varphi)\,d\varphi = \frac{a^2}{2}\cdot\frac{1}{2}\sin(2\varphi)\Big|_{\varphi=-\frac{\pi}{4}}^{\varphi=\frac{\pi}{4}} =$$

$$= \frac{a^2}{4}\left(1-(-1)\right) = \frac{a^2}{2}.$$

2. The length of a parametric curve is given by:

$$l = \int_{t_0}^{t_1} \|\dot{\vec{p}}(t)\|\,dt$$



In our case $\dot{\vec{p}}(t) = \begin{bmatrix} 2t\cos t - t^2\sin t \\ 2t\sin t + t^2\cos t \end{bmatrix}$,

and $\|\dot{\vec{p}}(t)\|^2 = (2t\cos t - t^2\sin t)^2 + (2t\sin t + t^2\cos t)^2 =$
$= \ldots = 4t^2 + t^4$. ("mixed" terms cancel)

Therefore: $l = \int_0^{2\pi} t\sqrt{t^2+4}\,dt \underset{\substack{u=t^2+4 \\ du = 2t\,dt}}{=} \int_4^{4\pi^2+4} \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\,\frac{2}{3}\,u^{3/2}\Big|_{u=4}^{u=4\pi^2+4} =$

$$= \frac{1}{3}\left(8\left(\pi^2+1\right)^{3/2} - 8\right) = \frac{8}{3}\left(\left(\pi^2+1\right)^{3/2} - 1\right).$$

3. We can evaluate length with the same formula as in the previous exercise.

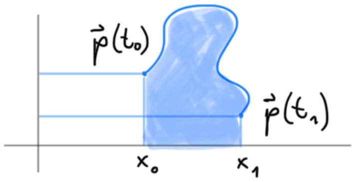
$$\dot{\vec{q}}(t) = \begin{bmatrix} 1-\cos t \\ \sin t \end{bmatrix}, \quad \|\dot{\vec{q}}(t)\|^2 = (1-\cos t)^2 + \sin^2 t = $$
$$= 1 - 2\cos t + \cos^2 t + \sin^2 t =$$
$$= 2 - 2\cos t.$$

Hence: $l = \int_0^{2\pi} \sqrt{2-2\cos t}\, dt = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt =$

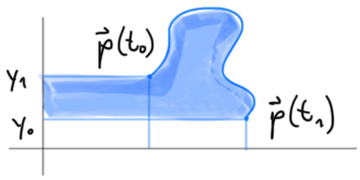
$$2 - 2\cos t = 4\,\frac{1-\cos t}{2} = 4\sin^2\left(\frac{t}{2}\right)$$

$$= 2\cdot 2\left(-\cos\left(\frac{t}{2}\right)\right)\Big|_{t=0}^{t=2\pi} = 4\cdot(1-(-1)) = 8$$

To evaluate the area, there are several formulas:



$$A = \int_{x_0}^{x_1} y\, dx = \int_{t_0}^{t_1} y(t)\,\dot{x}(t)\, dt$$



$$A = \int_{y_0}^{y_1} x\, dy = \int_{t_0}^{t_1} x(t)\,\dot{y}(t)\, dt$$

These two evaluate the area of the shaded region.

In our case we'll conveniently use the first one:

$$A = \int_0^{2\pi} \underbrace{(1-\cos t)}_{= y(t)}\underbrace{(1-\cos t)}_{= \dot{x}(t)}\, dt = \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt =$$

$$= 2\pi + 0 + \frac{1}{2}\cdot 2\pi = 3\pi.$$

5. (a) Plug the components of the parametrization directly into the implicit formula and check that the left-hand side simplifies to r^2.

(b) The normal to an implicit surface $f(x,y,z) = C$ is parallel to grad f at the given point. In our case:

$$f(x,y,z) = \left(2 - \sqrt{x^2+y^2}\right)^2 + z^2 \quad \text{and}$$

$$\operatorname{grad} f = \begin{bmatrix} 2\left(2-\sqrt{x^2+y^2}\right)\cdot\left(-\frac{2x}{2\sqrt{x^2+y^2}}\right) \\ \cdots \\ 2z \end{bmatrix} = \begin{bmatrix} 2\left(x - 2x/\sqrt{x^2+y^2}\right) \\ 2\left(y - 2y/\sqrt{x^2+y^2}\right) \\ 2z \end{bmatrix}.$$

At $T(1, \sqrt{3}, 1)$:

$(\operatorname{grad} f)(1, \sqrt{3}, 1) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, hence $z = 1$ is the sought tangent plane.