Mathematical modelling
Chapter 3
Dynamical modelling: differential equations

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2019/2020
Homogeneous DE

A homogeneous (nonlinear) DE is of the form

\[ \dot{x} = f\left(\frac{x}{t}\right). \]  \hspace{1cm} (1)

The solution is obtained by introducing a new dependent variable

\[ u = \frac{x}{t}. \]

Hence \( x = ut \) and differentiating with respect to \( t \) we get

\[ \dot{x} = \dot{u}t + u. \]  \hspace{1cm} (2)

Plugging (2) into (1) we get

\[ \dot{u}t + u = f(u). \]  \hspace{1cm} (3)

Rearranging (3) we obtain

\[ t\dot{u} = f(u) - u, \]

which is a separable DE.
Example - homogeneous DE

\[
y' = \frac{y - x}{x}
\]

can be written as

\[
y' = \frac{y}{x} - 1.
\] (4)

Introducing a new dependent variable

\[
u = \frac{y}{x},
\]

plugging in (4), we get

\[
u'x + u = u - 1.
\] (5)

This is equivalent to

\[
u'x = -1
\]

and hence

\[u = \frac{y}{x} = \log\left(\frac{C}{x}\right).\]
Orthogonal trajectories

Definition and the procedure for solving

Given a 1-parametric family of curves

\[ F(x, y, a) = 0 \quad \text{where} \quad a \in \mathbb{R}, \]

an **orthogonal trajectory** is a curve

\[ G(x, y) = 0 \]

that intersects each curve from the given family at a right angle.

Assume that \( F \) is a differentiable function.

1. The family \( F(x, y, a) = 0 \) is the general solution of a 1st order DE, that is obtained by differentiating the equation with respect to the independent variable (using implicit differentiation) and eliminating the parameter \( a \).
2. By substituting \( y' \) for \( -1/y' \) in the DE for the original family, we obtain a DE for curves with orthogonal tangents at every point of intersection.
3. The general solution to this equation is the family of orthogonal trajectories to the original equation.
Example - orthogonal trajectories to the family of circles

Let us find the orthogonal trajectories to the family of circles through the origin with centers on the $y$ axis:

$$x^2 + y^2 - 2ay = 0.$$  \hspace{1cm} (6)

Differentiating (6) w.r.t. the independent variable gives

$$2x + 2yy' - 2ay' = 0.$$ \hspace{1cm} (7)

Expressing $a$ from (7) gives

$$a = \frac{x}{y'} + y.$$ \hspace{1cm} (8)

Inserting (8) into (6) we obtain the DE for the given family

$$x^2 - y^2 - \frac{2xy}{y'} = 0.$$ \hspace{1cm} (9)

Next we express $y'$ from (9) and obtain

$$y' = \frac{2xy}{x^2 - y^2}.$$ \hspace{1cm} (10)
Example - orthogonal trajectories to the family of circles

The DE for orthogonal trajectories is obtained by substituting $y'$ for $-1/y'$ in (10) to obtain

$$- \frac{1}{y'} = \frac{2xy}{x^2 - y^2},$$

(11)

which is equivalent to

$$y' = -\frac{x^2 - y^2}{2xy}.$$  

(12)

(12) is a homogeneous DE:

$$y' = -\frac{x^2 - y^2}{2xy} = -\frac{x}{2y} + \frac{y}{2x}.$$

By introducing $y = ux$ we obtain

$$u'x + u = -\frac{1}{2u} + \frac{u}{2} \Rightarrow u'x = -\frac{1 + u^2}{2u} \Rightarrow \frac{2udu}{1 + u^2} = -\frac{dx}{x}$$

$$\Rightarrow \log (1 + u^2) = -\log x + \log C$$

$$\Rightarrow 1 + u^2 = \frac{C}{x},$$
Example - orthogonal trajectories to the family of circles

Plugging in $u = \frac{y}{x}$ again gives the general solution

$$x^2 + y^2 = Cx.$$ 

Orthogonal trajectories to circles through the origin with centers on the $y$ axis are circles through the origin with centers on the $x$ axis.

Both families together form an orthogonal net:
Exact DE's

Notice first that a 1st order DE

$$\dot{x} = f(t, x)$$

can be rewritten in the form

$$M(t, x)\,dt + N(t, x)\,dx = 0. \quad (13)$$

Recall that the differential of a function $u(t, x)$ is equal to

$$du = \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx = \left( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} \right) \cdot (dt, dx),$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^2$.

DE (13) is exact if there exists a differentiable function $u(t, x)$ such that

$$\frac{\partial u}{\partial t} = M(t, x) \quad \text{and} \quad \frac{\partial u}{\partial x} = N(t, x).$$

In this case, solutions of (13) are level curves of the function $u$:

$$u(t, x) = C, \quad \text{where} \quad C \in \mathbb{R}.$$
Recall from Calculus that if $u$ has continuous second order partial derivatives then

$$\frac{\partial u}{\partial x \partial t} = \frac{\partial u}{\partial t \partial x}.$$ 

This gives the following necessary condition for exact differential equations

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}. \tag{14}$$

If $M$ and $N$ are differentiable for every $(t, x) \in \mathbb{R}^2$, the condition (14) is also sufficient.

A potential function $u$ can be determined from the following equality

$$u(x, t) = \int M(t, x) \, dt + C(x) = \int N(t, x) \, dx + D(t),$$

where $C(x)$ and $D(t)$ are some functions.
The DE

\[ x + ye^{2xy} + xe^{2xy} y' = 0 \]  \hspace{1cm} (15)

can be rewritten as

\[ (x + ye^{2xy})dx + xe^{2xy} dy = 0. \]  \hspace{1cm} (16)

The equation is exact since

\[ \frac{\partial (x + ye^{2xy})}{\partial y} = \frac{\partial (xe^{2xy})}{\partial x} = (e^{2xy} + 2xye^{2xy}). \]

A potential function is equal to

\[ u(x, y) = \int (x + ye^{2xy}) \, dx = \frac{x^2}{2} + \frac{1}{2} e^{2xy} + C(y) \]

\[ = \int (xe^{2xy}) \, dy = \frac{1}{2} e^{2xy} + D(x), \]

Defining \( C(y) = 0 \) and \( D(x) = x^2/2 \), we get

\[ u(x, y) = \frac{x^2}{2} + \frac{1}{2} e^{2xy}. \]
The general solution is the family of level curves

\[ \frac{x^2}{2} + \frac{1}{2} e^{2xy} = E, \quad E \in \mathbb{R}. \]
Let $D \subset \mathbb{R}^2$ be the domain of the function $f(x, y)$. For each point $(x, y) \in D$ the DE

$$y' = f(x, y)$$

gives the value $y'$ of the coefficient of the tangent to the solution $y(x)$ through this specific point, that is, the direction in which the solution passes through the point.

All these directions together form the directional field of the equation.

A solution of the equation is represented by a curve $y = y(x)$ that follows the given directions at every point $x$, i.e., the coefficient of the tangent corresponds to the value $f(x, y(x))$.

The general solution to the equation is a family of curves, such that each of them follows the given direction.
Directional fields and solutions of

\[ y' = ky \quad \quad \quad \quad y' = ky(1 - y) \]
Theorem (Existence and uniqueness of solutions)  
If $f(x, y)$ is continuous and differentiable with respect to $y$ on the rectangle

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b], \quad a, b > 0$$

then the DE with initial condition

$$y' = f(x, y), \quad y(x_0) = y_0,$$

has a unique solution $y(x)$ defined at least on the interval

$$[x_0 - \alpha, x_0 + \alpha], \quad \alpha = \min \left\{ a, \frac{b}{M}, \frac{1}{N} \right\},$$

where

$$M = \max\{f(x, y) : (x, y) \in D\} \quad \text{and} \quad N = \max\left\{ \frac{\partial f(x, y)}{\partial y} : (x, y) \in D \right\}.$$
Numerical methods for solving DE’s

We are given the DE with the initial condition

\[ y'(x) = f(y, x), \quad y(x_0) = y_0. \]

Instead of analytically finding the solution \( y(x) \), we construct a recursive sequence of points

\[ x_i = x_0 + ih, \quad y_i = y(x_i), \quad i \geq 0 \]

where \( y_i \) is an approximation to the value of the exact solution \( y(x_i) \), and \( h \) is the \textit{step size}.

A number of numerical methods exists, the choice depends on the type of equation, desired accuracy, computational time,...

We will look at the simplest and best known.
Euler’s method

*Euler’s method* is the simplest and most intuitive approach to numerically solve a DE.

At each step the value $y_{i+1}$ is obtained as the point on the tangent to the solution through $(x_i, y_i)$ at $x_{i+1} = x_i + h$:

- initial condition: $(x_0, y_0)$
- for each $i$: $x_{i+1} = x_i + h$, $y_{i+1} = y_i + hf(x_i, y_i)$.

The point $(x_{i+1}, y_{i+1})$ typically lies on a different particular solution than $(x_i, y_i)$, at each step, the error is of order $O(h^2)$. 
The **Runge-Kutta** method is probably the most widely used numerical method for DE’s:

- **initial condition**: \((x_0, y_0)\)
- at each step \(x_i\) is computed as a weighted average of approximations at \(x = x_i, x = x_i + h/2\) and \(x = x_{i+1}\):

\[
x_{i+1} = x_i + h, \quad y_{i+1} = y_i + \left( \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \right),
\]

- \(k_1 = h \cdot f(x_i, y_i),\)
- \(k_2 = h \cdot f(x_i + h/2, y_i + k_1/2),\)
- \(k_3 = h \cdot f(x_i + h/2, y_i + k_2/2)\) in
- \(k_4 = h \cdot f(x_i + h, y_i + k_3)\)

The error at each step is of order \(O(h^5)\). The cumulative error is of order \(O(h^4)\).
Below is a comparison of Euler’s and Runge-Kutta methods for the DE

\[ y' = -y - 1, \quad y(0) = 1 \]

with step size \( h = 0.3 \):

The red curve is the exact solution \( y = 2e^{-x} - 1 \).