1. [10 points] Let $A, B$ be $m \times n$ matrices, $m, n \in \mathbb{N}$, such that $A^T B = 0$ and $B A^T = 0$. Verify the following statements:

(a) [1] Every column of $A$ is perpendicular to every column of $B$.

*Hint:* What is the meaning of the entry in the $i$-th row and $j$-th column of $A^T B$?

*Solution.*

$$0 = (A^T B)_{i,j} = \text{ dot product of the } i\text{-th row of } A^T \text{ and } j\text{-th column of } B$$

(b) [2] $A^+ B = B^+ A = 0$.

*Hint:* Remember the geometric meaning of $A^+ b$ (resp. $B^+ a$), where $b$ (resp. $a$) is a column in $\mathbb{R}^m$, and use this for every column of the matrix $B$ (resp. $A$).

*Solution.* $A^+ b$ is the vector with the smallest norm among all vectors from the set 

$$S(A, b) := \{ x \in \mathbb{R}^n : \| b - A x \| = \min_{x' \in \mathbb{R}^n} \| b - A x' \| \}.$$ 

Since every column $b_j$ of $B$ is perpendicular to the span of the columns of $A$, $\min_{x' \in \mathbb{R}^n} \| b_j - A x' \| = \| b_j \|$ and hence $0 \in S(A, b_j)$. Thus $0 = A^+ b_j$.

(c) [1] Every column of $A^T$ is perpendicular to every column of $B^T$.

*Hint:* What is the meaning of the entry in the $i$-th row and $j$-th column of $(B^T)^T A^T = B A^T$?

*Solution.*

$$0 = (B A^T)_{i,j} = \text{ dot product of the } i\text{-th row of } B \text{ and } j\text{-th column of } A^T$$

(d) [2] $B A^+ = A B^+ = 0$.

*Hint:* Assuming (1b) is true, this statement can be proved by plugging $A^T$ and $B^T$ into the appropriate variables in (1b).

*Solution.* Plugging $A^T, B^T$ into $A, B$ of (1b) we obtain

$$0 = (A^T)^+ B^T = (A^+)^T B^T = (B A^+)^T \Rightarrow 0 = B A^+,$$

$$0 = (B^T)^+ A^T = (B^+)^T A^T = (A B^+)^T \Rightarrow 0 = A B^+.$$ 

(e) [4] $(A + B)^+ = A^+ + B^+$.

*Hint:* Use (1b), (1d) in the verification of this part.
2. \[10\text{ points}\] For the parametric curve
\[
f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2t - t^2 \\ 3t - t^3 \end{bmatrix}, \quad \text{where} \quad t \in \mathbb{R},
\]
solve the following:

(a) \[1\text{ point}\] Find intersections with coordinate axes.
\[\text{Solution.}\]
\[
x(t) = 0 \iff 2t - t^2 = 0 \iff t(t - 2) = 0 \iff t \in \{0, 2\},
\]
\[
y(t) = 0 \iff 3t - t^3 = 0 \iff t(3 - t^2) = 0 \iff t \in \{-\sqrt{3}, 0, \sqrt{3}\}.
\]
Intersections with \(y\)-axis: (0, 0), (0, -2).
Intersections with \(x\)-axis: (-2\sqrt{3} - 3, 0), (0, 0), (2\sqrt{3} - 3, 0).

(b) \[1\text{ point}\] Find points at which the tangent is horizontal or vertical.
\[\text{Solution.}\]
\[
\dot{x}(t) = 0 \iff 2 - 2t = 0 \iff t = 1,
\]
\[
\dot{y}(t) = 0 \iff 3 - 3t^2 = 0 \iff 1 - t^2 = 0 \iff t \in \{-1, 1\}.
\]
Horizontal tangent: (-3, -2), vertical tangents: none.

(c) \[1\text{ point}\] Find points where \(x'(t) = y'(t) = 0\).
\[\text{From the part above (1, 2),}\]

(d) \[1\text{ point}\] Determine the asymptotic behaviour (limits as \(t \to \pm \infty\)).
\[\text{Solution.} \quad \lim_{t \to -\infty} f(t) = \begin{bmatrix} -\infty \\ \infty \end{bmatrix} \quad \lim_{t \to \infty} f(t) = \begin{bmatrix} -\infty \\ -\infty \end{bmatrix}.
\]

(e) \[2\text{ points}\] Show that there are no self-intersections.
\[\text{Hint:} \quad \text{To notice that the curve does not have any self-intersections verify that} \quad 1 - x(t) = 1 - x(s) \text{ implies} \quad s = 2 - t \text{ and plug this into the equation} \quad y(t) = y(s).
\]
\[\text{Solution.} \quad \text{Assume that} \quad t \neq s.
\]
\[
1 - x(t) = 1 - x(s) \iff 1 - 2t + t^2 = 1 - 2s + s^2 \iff (1 - t)^2 = (1 - s)^2 \iff 1 - t \in \{1 - s, s - 1\}.
\]
Since \(t \neq s\), \(1 - t = s - 1\) and hence \(s = 2 - t\). Thus
\[
y(t) = y(2 - t) \iff 3t - t^3 = 3(2 - t) + (2 - t)^3 \iff t^3 - 3t^2 + 3t - 1 = 0 \iff (t - 1)^3 = 0 \iff t = 1.
\]
But then \(s = 2 - 1 = 1\) and \(s = t\).

Note: If you are not able to prove (1b), you can assume it is true in proving (1d), and also you can assume both of them are true in proving (1e).
(f) [4] Plot the curve.

*Solution.* Using the information above, the sketch of the curve is the following:

![Plot of the curve](image)

3. [10 points] Let

\[
F(x, y) := \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} x^2 + y^2 - 10x + y \\ x^2 - y^2 - x + 10y \end{bmatrix}
\]

be a vector function and \(a = (2, 4) \in \mathbb{R}^2\) a point.

(a) [2] Calculate the Jacobian matrix of the function \(F\) in the point \(a\).

*Solution.*

\[
JF(a) = \begin{bmatrix} 2x - 10 \\ 2x - 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.
\]

(b) [3] Calculate the linear approximation of \(F\) in the point \(a\).

*Solution.*

\[
L_{F,a}(x, y) = F(a) + JF(a)(x - 2, y - 4) = \begin{bmatrix} 4 \\ 26 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix} \begin{bmatrix} x - 2 \\ y - 4 \end{bmatrix} = \begin{bmatrix} -6x + 9y - 20 \\ 3x + 2y + 12 \end{bmatrix}.
\]

(c) [5] Perform one step of Newton’s method to find the approximate solution of the system

\[
F(x, y) = \begin{bmatrix} 1 \\ 25 \end{bmatrix}
\]

with the initial approximation \(a\).

*Solution.* We are searching for zeroes of the vector function

\[
G(x, y) = F(x, y) - \begin{bmatrix} 1 \\ 25 \end{bmatrix}
\]

with the initial approximation \((x_0, y_0) = a\). One step of Newton’s method:

\[
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + JG(a)^{-1}G(x_0, y_0)
\]

\[
= \begin{bmatrix} 4 \\ 26 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \frac{1}{39} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 25 \\ 47 \end{bmatrix}.
\]
4. [10 points] Find the solution \([x(t), y(t)]\) of the nonautonomous system of first order linear equations

\[
\begin{align*}
\dot{x} &= 2x - y, \\
\dot{y} &= -2x + y + 18t,
\end{align*}
\]

which satisfies \(x(0) = 1, y(0) = 0\).

*Hint:* One of the particular solutions of the system above is of the form \(x(t) = At^2 + Bt + C, y(t) = Dt^2 + Et + F\), where \(A, B, C, D, E, F\) are constants.

**Solution.** The system in the matricial form

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} =
\begin{bmatrix}
2 & -1 \\
-2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} +
\begin{bmatrix}
0 \\
18t
\end{bmatrix} =: A
\begin{bmatrix}
x \\
y
\end{bmatrix} + f(t).
\]

Solution of the homogeneous part \((x_h(t), y_h(t))\):

\[
\det(A - \lambda I) = \det \begin{bmatrix}
2 - \lambda & -1 \\
-2 & 1 - \lambda
\end{bmatrix} = (2 - \lambda)(1 - \lambda) - 2 = \lambda(\lambda - 3).
\]

Hence \(\det(A - \lambda I) = 0\) for \(\lambda_1 = 0\) and \(\lambda_2 = 3\). Further on,

\[
\ker A = \ker \begin{bmatrix}
2 & -1 \\
-2 & 1
\end{bmatrix} = \ker \begin{bmatrix}
2 & -1 \\
0 & 0
\end{bmatrix} = \mathcal{L}\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \},
\]

\[
\ker(A - 3I) = \ker \begin{bmatrix}
-1 & -1 \\
-2 & -2
\end{bmatrix} = \ker \begin{bmatrix}
-1 & -1 \\
0 & 0
\end{bmatrix} = \mathcal{L}\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}.
\]

Thus:

\[
\begin{bmatrix}
x_h(t) \\
y_h(t)
\end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

Particular solution \((x_p(t), y_p(t))\) using the hint:

\[
\begin{align*}
2At + B &= 2(At^2 + Bt + C) - (Dt^2 + Et + F) = (2A - D)t^2 + (2B - E)t + (2C - F), \\
2Dt + E &= -2(At^2 + Bt + C) + (Dt^2 + Et + F) + 18t = (-2A + D)t^2 + (-2B + E + 18)t + (-2C + F).
\end{align*}
\]

By comparing the coefficients at 1, t, t^2 we get the system:

\[
2A - D = 0, \quad 2A = 2B - E, \quad B = 2C - F, \quad 2D = -2B + E + 18, \quad E = 2C - F,
\]

with a one parametric solution

\[
(A, B, C, D, E, F) = (3, 2, C, 6, -2, -2 + 2C).
\]

Choosing \(C = 0\) we get

\[
(x_p(t), y_p(t)) = (3t^2 + 2t, 6t^2 + 2t - 2).
\]

Finally, a general solution is

\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3t^2 + 2t \\
6t^2 + 2t - 2
\end{bmatrix}.
\]

The one satisfying \(x(0) = 1, y(0) = 0\) has \(\alpha = 1, \beta = 0\).