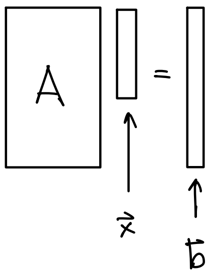
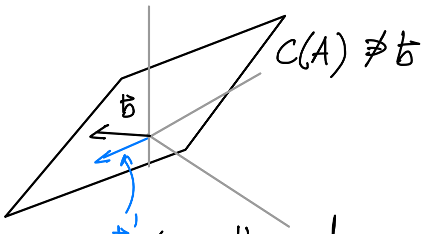
Matematično modeliranje (FRI), vaje, 23.02.2021

How to solve $A\vec{x} = \vec{b}$ if it has more equations than unknowns?



In general (and usually) there are no solutions.

Column space of A is a vector subspace in $\mathbb{R}^n$ (for some n).



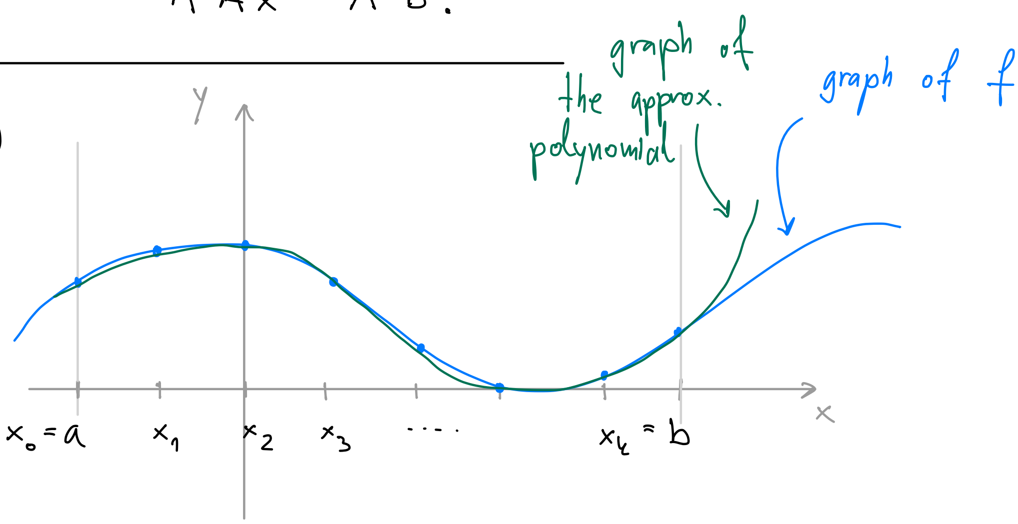
$\vec{b}'$ ← orthogonal projection of $\vec{b}$ onto $C(A)$

Instead of solving $A\vec{x} = \vec{b}$, we solve $A\vec{x} = \vec{b}'$, the solution $\vec{x}$ of $A\vec{x} = \vec{b}'$ is called the linear least squares solution of $A\vec{x} = \vec{b}$.

This $\vec{x}$ is exactly the solution of the associated normal equations:

$$A^T A \vec{x} = A^T \vec{b}.$$

1.)



(a) $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

$p(x_i) = f(x_i)$ ... $a_n x_i^n + a_{n-1} x_i^{n-1} + \ldots + a_1 x_i + a_0 = f(x_i)$

for each $i = 0, \ldots, k$

$$i=0: \quad x_0^n \cdot a_n + x_0^{n-1} \cdot a_{n-1} + \dots + x_0 \cdot a_1 + a_0 = f(x_0)$$

$$i=1: \quad x_1^n \cdot a_n + x_1^{n-1} \cdot a_{n-1} + \dots + x_1 a_1 + a_0 = f(x_1)$$

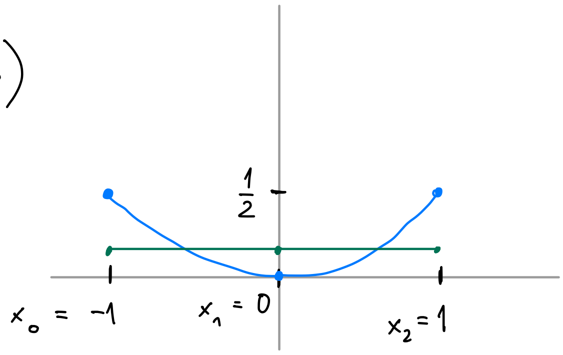
$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$i=k: \quad x_k^n \cdot a_n + x_k^{n-1} \cdot a_{n-1} + \dots + x_k a_1 + a_0 = f(x_k)$$

$$A = \begin{bmatrix} x_0^n & x_0^{n-1} & \dots & x_0 & 1 \\ x_1^n & x_1^{n-1} & \dots & x_1 & 1 \\ \vdots & & & \vdots & \vdots \\ x_k^n & x_k^{n-1} & \dots & x_k & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_k) \end{bmatrix}.$$

(b)

$$f(x) = \frac{x^2}{1+x^2}$$



- Degree 0 polynomial approximation: $p(x) = a_0$

In this case $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x} = a_0$, $\vec{b} = \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$

$a_0 = 1/2$
$a_0 = 0$
$a_0 = 1/2$

The corresponding normal system $A^T A \vec{x} = A^T \vec{b}$ is:

$$\underline{A^T A} = [1, 1, 1] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3, \qquad \underline{A^T \vec{b}} = [1, 1, 1] \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = 1$$

$\underline{3} a_0 = \underline{1} \quad \dots \quad a_0 = \frac{1}{3} \leftarrow$ *the linear least squares solution to $A\vec{x} = \vec{b}$.*

$$p(x) = \frac{1}{3}.$$

- Degree 1 polynomial approximation: $p(x) = a_1 x + a_0$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

Corresponding normal system:

$$A^T A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad A^T \vec{b} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}}_{= \begin{bmatrix} 2a_1 \\ 3a_0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{matrix} \dots & a_1 = 0 \\ \dots & a_0 = 1/3 \end{matrix} \qquad \underline{p(x)} = 0 \cdot x + \frac{1}{3} = \underline{\frac{1}{3}}.$$

– Degree 2 approximation : $p(x) = a_2 x^2 + a_1 x + a_0$

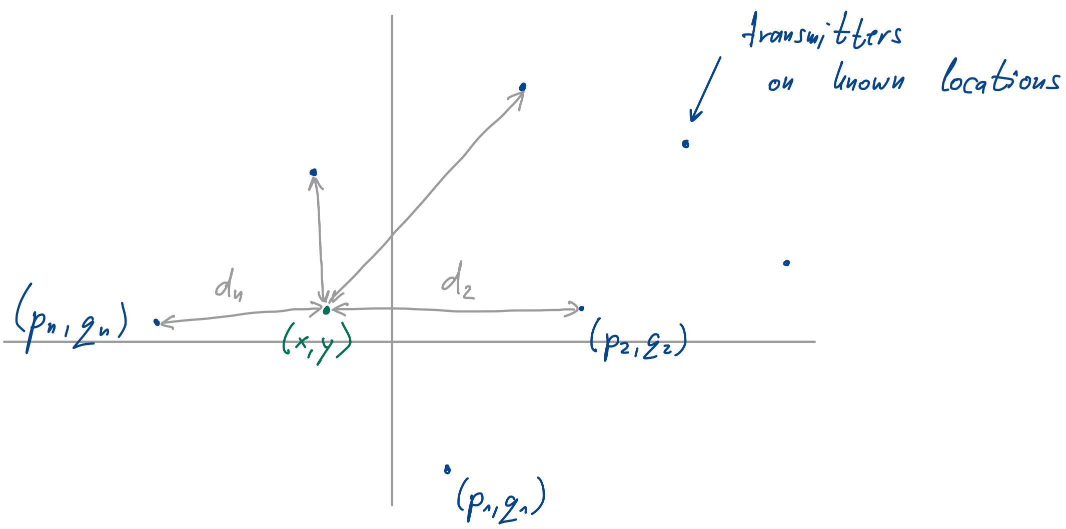
$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} a_2 \\ a_1 \\ a_1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

We have as many equation as unknowns – no need to solve the normal system, we solve the original one!

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1/2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1/2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \begin{matrix} a_2 = \frac{1}{2} \\ a_1 = 0 \\ a_0 = 0 \end{matrix}$$

$$\underline{p(x) = \frac{1}{2} x^2}.$$

2.



Determine the position (x,y) of the receiver knowing the distances to and positions of those transmitters.

for i-th transmitter we get: $(x-p_i)^2 + (y-q_i)^2 = d_i^2$

$$
\begin{aligned}
(x-p_1)^2 + (y-q_1)^2 &= d_1^2 \quad \ldots \quad x^2 - 2xp_1 + p_1^2 + y^2 - 2yq_1 + q_1^2 = d_1^2 \\
(x-p_2)^2 + (y-q_2)^2 &= d_2^2 \quad \ldots \quad x^2 - 2xp_2 + p_2^2 + y^2 - 2yq_2 + q_2^2 = d_2^2 \\
&\vdots
\end{aligned}
\Bigg\} \; -
$$

$$2x(p_2 - p_1) + p_1^2 - p_2^2 + 2y(q_2 - q_1) + q_1^2 - q_2^2 = d_1^2 - d_2^2$$

$$2(p_2 - p_1)x + 2(q_2 - q_1)y = d_1^2 - d_2^2 + p_2^2 - p_1^2 + q_2^2 - q_1^2$$

We can do this for each two consecutive equations.

From n non-linear equations we obtain $n-1$ linear equations.