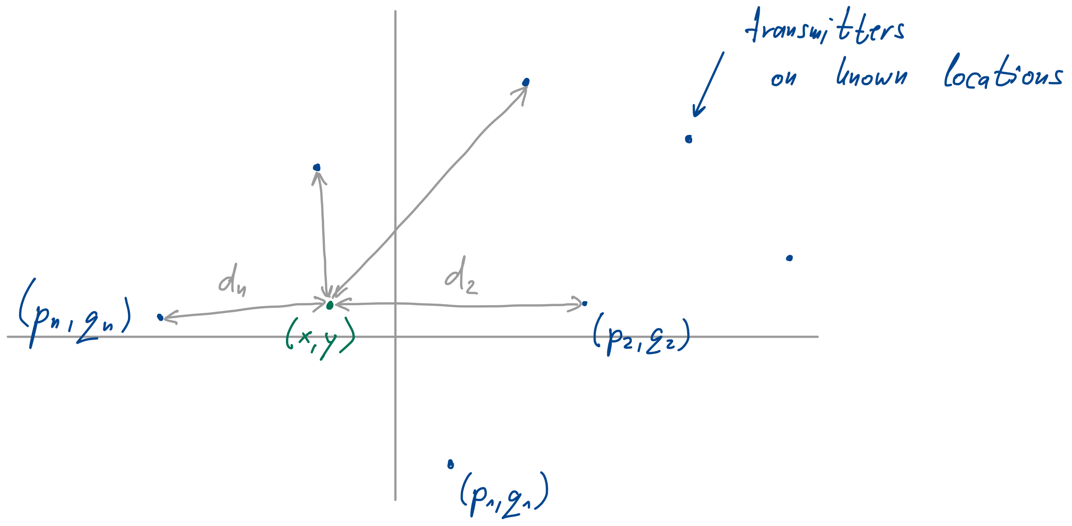
Matematično modeliranje (FRI), vaje, 02.03.2021

2.



Determine the position (x,y) of the receiver knowing the distances to and positions of those transmitters.

for i-th transmitter we get: $(x-p_i)^2 + (y-q_i)^2 = d_i^2$

(a)

$$(x-p_1)^2 + (y-q_1)^2 = d_1^2 \quad \dots \quad x^2 - 2xp_1 + p_1^2 + y^2 - 2yq_1 + q_1^2 = d_1^2$$

$$(x-p_2)^2 + (y-q_2)^2 = d_2^2 \quad \dots \quad x^2 - 2xp_2 + p_2^2 + y^2 - 2yq_2 + q_2^2 = d_2^2$$

$$\vdots$$

$$2x(p_2-p_1) + p_1^2 - p_2^2 + 2y(q_2-q_1) + q_1^2 - q_2^2 = d_1^2 - d_2^2$$

$$2(p_2-p_1)x + 2(q_2-q_1)y = d_1^2 - d_2^2 + p_2^2 - p_1^2 + q_2^2 - q_1^2$$

We can do this for each two consecutive equations.

From n non-linear equations we obtain $n-1$ linear equations:

For ith and $(i+1)^{st}$ equations we have:

$$x^2 - 2xp_i + p_i^2 + y^2 - 2yq_i + q_i^2 = d_i^2$$

$$x^2 + 2xp_{i+1} + p_{i+1}^2 + y^2 - 2yq_{i+1} + q_{i+1}^2 = d_{i+1}^2$$

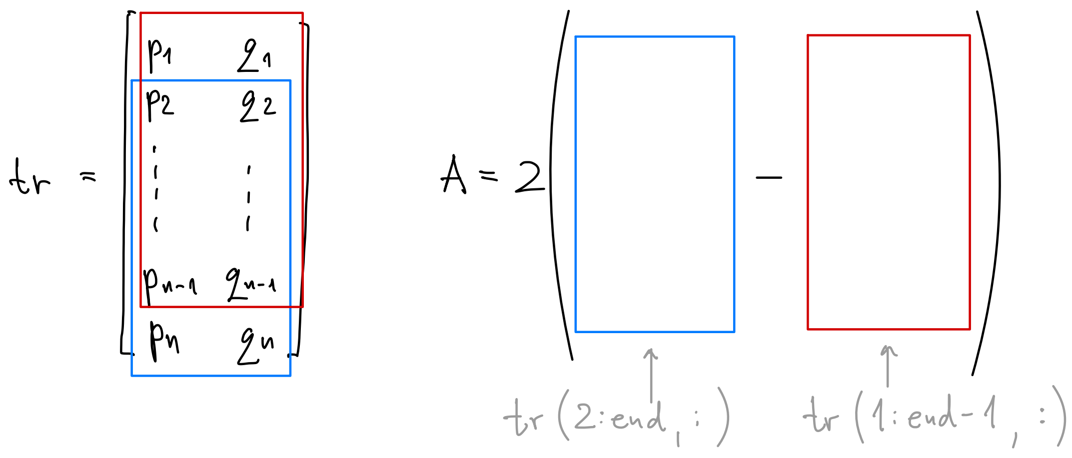
subtract

$$2(p_{i+1}-p_i)x + 2(q_{i+1}-q_i)y = d_i^2 - d_{i+1}^2 + p_{i+1}^2 - p_i^2 + q_{i+1}^2 - q_i^2, \quad i = 1, \dots, n-1$$

(b) Vector of unknowns $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

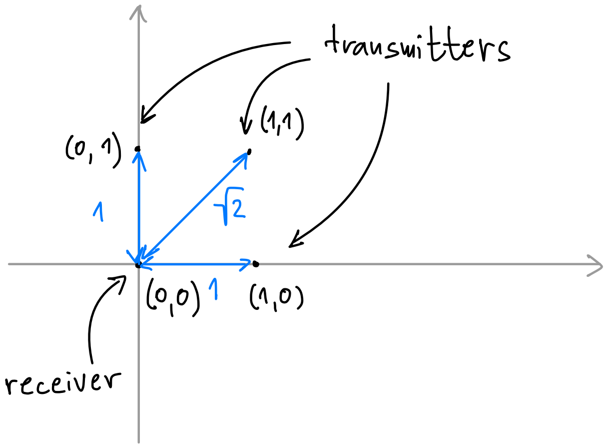
$$A = \begin{bmatrix} 2(p_2 - p_1), & 2(q_2 - q_1) \\ \vdots & \vdots \\ 2(p_{i+1} - p_i), & 2(q_{i+1} - q_i) \\ \vdots & \\ 2(p_n - p_{n-1}), & 2(q_n - q_{n-1}) \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} d_1^2 - d_2^2 + p_2^2 - p_1^2 + q_2^2 - q_1^2 \\ \vdots \\ d_i^2 - d_{i+1}^2 + p_{i+1}^2 - p_i^2 + q_{i+1}^2 - q_i^2 \\ \vdots \\ d_{n-1}^2 - d_n^2 + p_n^2 - p_{n-1}^2 + q_n^2 - q_{n-1}^2 \end{bmatrix}.$$

(c)



... something in a similar fashion for the right-hand side...

For the built-in test we use an artificial case with known receiver position, e.g.:

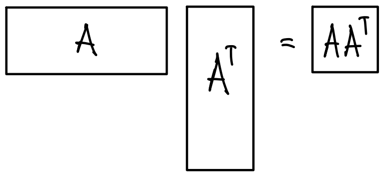


1. (a) $x - y + z - w = 1$
$x + y - z - w = 3$

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

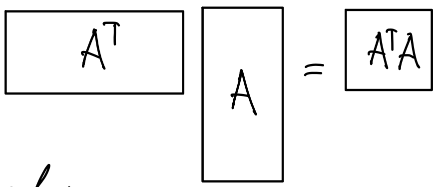
(b) Let's find the Moore-Penrose inverse of A:

$$A^+ = A^T (A A^T)^{-1},$$



if A has full rank and more columns than rows

($A^+ = (A^T A)^{-1} A^T$



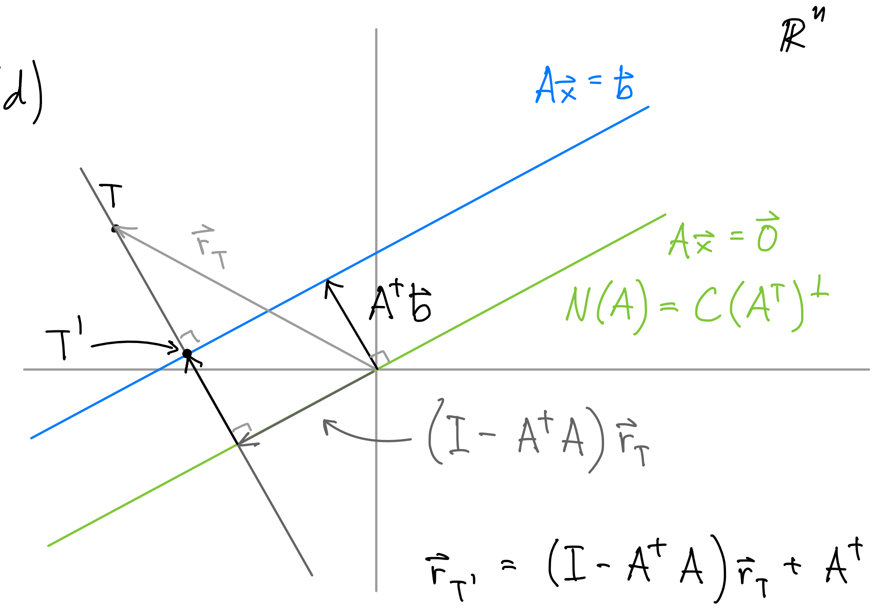
if A has full rank and more rows than columns)

$$A A^T = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}}_{4I} \quad \dots \quad (AA^T)^{-1} = \frac{1}{4} I = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

Hence:

$$\underline{A^+} = A^T \cdot (AA^T)^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix} \cdot \frac{1}{4} I = \underline{\frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}}.$$

(d)



$$\vec{r}_{T'} = (I - A^+A)\vec{r}_T + A^+\vec{b} =$$

$$= \vec{r}_T - A^+A\vec{r}_T + A^+\vec{b} = \underline{\vec{r}_T + A^+(\vec{b} - A\vec{r}_T)}. \quad (*)$$

In our case: $\vec{r}_T = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$, $A^+ = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$,

plug this into (*):

$$\vec{r}_{T'} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix} + \underbrace{\frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix} \underbrace{\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right)}_{\begin{bmatrix} 3 \\ 5 \end{bmatrix}}}_{\begin{bmatrix} 8 \\ 2 \\ -2 \\ -8 \end{bmatrix}} = \begin{bmatrix} 2 \\ -1/2 \\ -3/2 \\ 0 \end{bmatrix}, \quad \underline{T'\left(2, -\tfrac{1}{2}, -\tfrac{3}{2}, 0\right).}$$