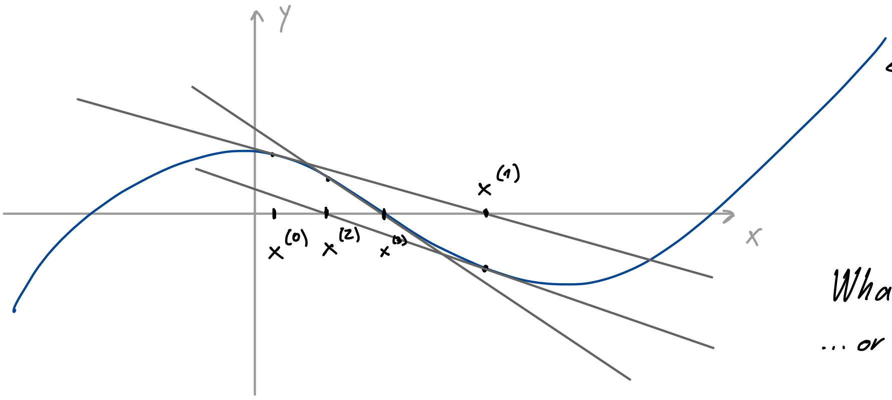
Matematično modeliranje (FRI), vaje, 16. 03. 2021



← graph of a function of a single variable x,

$$y = f(x)$$

What are zeros of f?
... or (at least) one zero of f?

$$f(x) = 0.$$

Newton's method to determine one zero of f:

start with an initial guess $x^{(0)}$,

evaluate $$x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})}$$

repeat $$x^{(2)} = x^{(1)} - \frac{f(x^{(1)})}{f'(x^{(1)})}$$

$$\vdots$$

For a vector-valued function $\vec{F} : U \longrightarrow \mathbb{R}^n$, $U \subseteq \mathbb{R}^n$, we replace f' by the Jacobi matrix of $\vec{F} = [f_1, f_2, \dots, f_n]^T$

$$J\vec{F} = \left[\frac{\partial f_i}{\partial x_j}\right]_{i,j=1}^{n}$$

A general step of Newton's method looks like:

$$\vec{x}^{(n+1)} = \vec{x}^{(n)} - \left[J\vec{F}\left(\vec{x}^{(n)}\right)\right]^{-1} \vec{F}\left(\vec{x}^{(n)}\right)$$

Instead we solve $J\vec{F}(\vec{x}^{(n)})\,\vec{y} = \vec{F}(\vec{x}^{(n)})$.

This (in nice circumstances) converges to one solution of $\vec{F}(\vec{x}) = \vec{0}$.

1.

$$x_1^2 - x_2^2 = 1,$$
$$x_1 + x_2 - x_1 x_2 = 1,$$

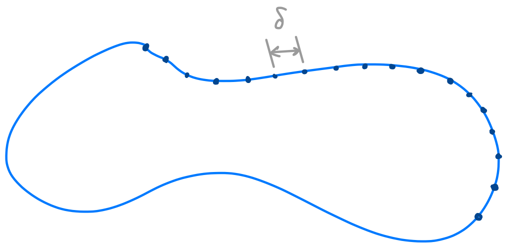
$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\quad \vec{F}(\vec{x}) = \begin{bmatrix} x_1^2 - x_2^2 - 1 \\ x_1 + x_2 - x_1 x_2 - 1 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$

$$x_1^2 - x_2^2 - 1 = 0$$
$$x_1 + x_2 - x_1 x_2 - 1 = 0$$

$$\vec{F}(\vec{x}) = \vec{0}$$

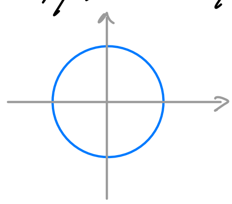
$$J\vec{F} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & -2x_2 \\ 1 - x_2 & 1 - x_1 \end{bmatrix}$$

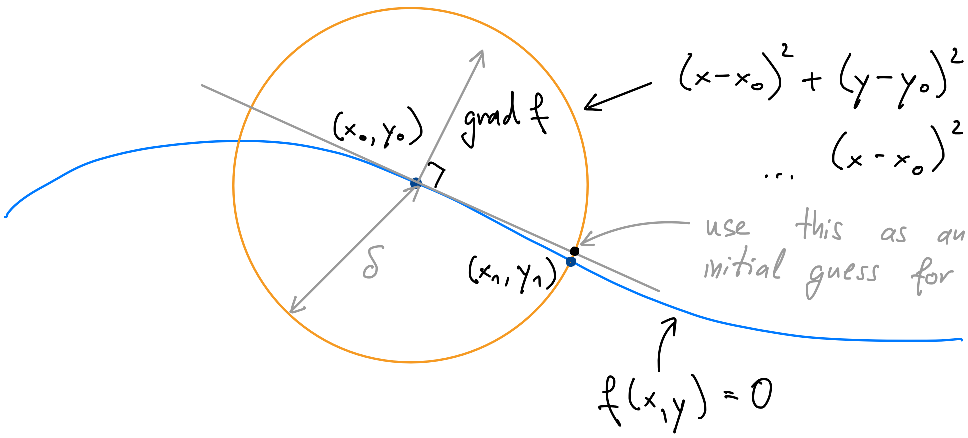
2. $f(x,y) = 0 \leftarrow$ this is a curve in $\mathbb{R}^2$ (the level set of f at 0)



(A known example $f(x,y) = x^2 + y^2 - 1$

... $x^2 + y^2 = 1$ ...)





$(x - x_0)^2 + (y - y_0)^2 = \delta^2$ ...

... $(x - x_0)^2 + (y - y_0)^2 - \delta^2 = 0$

use this as an initial guess for Newton's method

Hence $\vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x,y) \\ (x - x_0)^2 + (y - y_0)^2 - \delta^2 \end{bmatrix}$

$$J\vec{F} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ 2(x - x_0) & 2(y - y_0) \end{bmatrix} \qquad \text{grad } f = \begin{bmatrix} a \\ b \end{bmatrix}$$

For the initial guess we use $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} -b \\ a \end{bmatrix} \cdot \dfrac{\delta}{\left\| \begin{bmatrix} -b \\ a \end{bmatrix} \right\|}$.