Mathematical modelling

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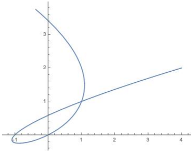
Problem: find the self-intersection (if there is one) of a parametric curve

Let $f(t) = \begin{bmatrix} t^3 - 2t \\ t^2 - t \end{bmatrix}$



A self-intersection is at a point $f(t) = f(s)$, with $t \neq s$, so:

$$t^3 - 2t = s^3 - 2s, \qquad t^2 - t = s^2 - s$$
$$t^3 - s^3 = 2t - 2s, \qquad t^2 - s^2 = t - s$$

Since $t \neq s$ we can divide by $t - s$:

$$t^2 + ts + s^2 = 2, \qquad t + s = 1,$$
$$t = 1 - s, \qquad (1-s)^2 + s(1-s) + s^2 = 2.$$

The self-intersection (where s and t can be interchanged) is at

$$s = (1+\sqrt{5})/2, t = (1-\sqrt{5})/2, \quad f(t) = f(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem: do two parametric curves intersect. Imagine two cars speeding along the two curves. Do they crash?

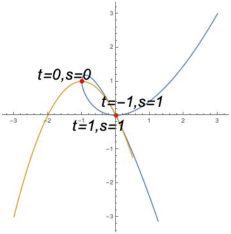
Let $f_1(t) = \begin{bmatrix} t^2 - 1 \\ -t^3 - t^2 + t + 1 \end{bmatrix}, \quad f_2(s) = \begin{bmatrix} s - 1 \\ 1 - s^2 \end{bmatrix}.$

To find the intersections, solve the system

$$\begin{aligned} t^2 - 1 &= s - 1, \qquad & -t^3 - t^2 + t + 1 &= 1 - s^2 \\ s &= t^2 & -s^6 - s^4 + s^2 + 1 &= 1 - s^2 \end{aligned}$$

There are three solutions (work them out!!):

$$\begin{aligned} t = -1, s = 1 &\quad \Rightarrow \ x = 0, y = 0 \\ t = 0, s = 0 &\quad \Rightarrow x = -1, y = 1 \\ t = 1, s = 1 &\quad \Rightarrow x = 0, y = 0 \end{aligned}$$

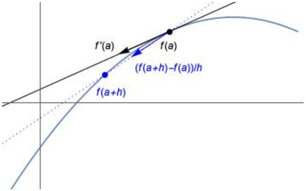


The superheroes meet at $t = 0, s = 0$ at the point $(-1, 1)$ and at $t = 1, s = 1$ at the point $(0, 0)$.

Derivative, linear approximation, tangent

The *derivative* of the vector function $f(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix}$ at the point a is the vector:

$$Df(a) = \begin{bmatrix} x_1'(a) \\ \vdots \\ x_m'(a) \end{bmatrix} = f'(a) = \lim_{h \to 0} \frac{1}{h}(f(a+h) - f(a))$$

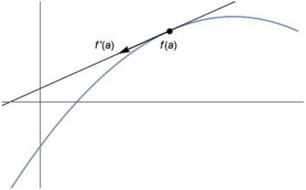


The vector $f'(a)$ (if it exists) represents the *velocity vector* of a point moving along the curve at the point $t = a$.

If $f'(a) \neq 0$ it points in the direction of the tangent at $t = a$.

The *linear approximation* of f at $t = a$ is

$$L_a(t) = f(a) + (t - a)f'(a)$$



If $f'(a) \neq \mathbf{0}$, this is a parametric line corresponding to the tangent line to the curve $f(t)$ at $t = a$.

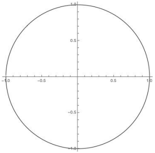
In this case $f(a)$ is a *regular point* of the parametric curve and the parametric curve is *smooth* at $t = a$.

If $f'(a) = \mathbf{0}$ (or if it does not exist), the point $f(a)$ is *singular*.

A curve $C \in \mathbb{R}^m$ is *smooth* at a point x on C if there exists a parametrization $f(t)$ of C, such that $f(a) = x$ and $f'(a) \neq 0$.

A smooth curve has a tangent at every point $x \in C$.

Problem: Is the curve $C = \{f(t), t \in [0, \sqrt{2\pi}]\}$, $f(t) = \begin{bmatrix} \cos(t^2) \\ \sin(t^2) \end{bmatrix}$, smooth?



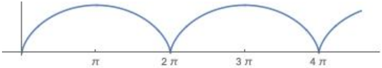
Since $x^2 + y^2 = 1$, $f(t)$ is a parametrization of the unit circle which is a smooth curve (it has a tangent at every point).

Since $f'(0) = \mathbf{0}$ the parametrization $f(t)$ is not a smooth at $t = 0$.

Find a smooth parametrization!

Problem: is the cycloid a smooth curve?

Our parametrization



$$f(t) = \begin{bmatrix} (t - \sin t) \\ (1 - \cos t) \end{bmatrix}, \quad f'(t) = \begin{bmatrix} (1 - \cos t) \\ \sin t \end{bmatrix}$$

is not smooth at $t = 2k\pi$ since $f'(2k\pi) = \mathbf{0}$.

Does a tangent exist? (It seems not, but let's check...)

The slope of the tangent line at a point $f(t)$ is:

$$k_t = \frac{y'(t)}{x'(t)} = \frac{a \sin t}{a(1 - \cos t)}$$

The left and right limits as $t \to 2k\pi$ are

$$\lim_{t \nearrow 2k\pi} k_t = \lim_{t \nearrow 2k\pi} \frac{\cos t}{\sin t} = -\infty, \quad lim_{t \searrow 2k\pi} k_t = \lim_{t \searrow 2k\pi} \frac{\cos t}{\sin t} = \infty,$$

so at these points the curve forms a sharp spike (a *cusp*) and a tangent does not exist.

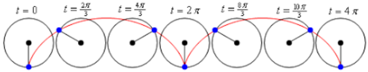
So, the cycloid is not smooth at the points where it touches the x axis.

(l'Hospital's rule was used to compute the limits.)

Arc length and the natural parametrization

The *arc length* s of a parametric curve $f(t)$, $t \in [a, b]$, in $\mathbb{R}^m$ is the length of the curve between the points $t = a$ in $t = b$, i.e. the distance covered by a point moving along the curve between these two points.

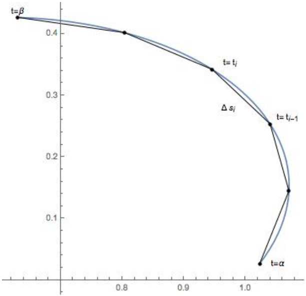
For example, what distance does a point on the circle cover when the circle makes one full turn?



How do we compute it?

An aproximate value for s is the length of a polygonal curve connecting close enough points on the curve:

$$s_n = \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|$$



For n big enough, s_n is a practical approximation for s.

If the function $f(t)$ is continuously differentiable, then we can approximate the value $f(t_i) = f(t_{i-1} + \Delta t)$, where $\Delta t = t_i - t_{i-1}$, by the linear approximation:

$$f(t_i) = f(t_{i-1}) + f'(t_{i-1})\Delta t$$

and we get:

$$\|f(t_i) - f(t_{i-1})\| \doteq \|f'(t_{i-1})\|\Delta t$$

$$s_n \doteq \sum_{i=1}^{n} \|f'(t_{i-1})\|\Delta t.$$

This is a Riemann integral sum of the function $\|f'(t)\|$.

In the limit as $n \to \infty$, s_n converges to s and the integral sum to the integral, so

$$s = \lim_{n\to\infty} s_n = \int_a^b \|f'(t)\|\, dt$$

Problem: The length of the path traced by a point on the circle after a full turn,

that is, of a cycle of the cycloid $f(t) = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix}$:

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt \\ &= \int_0^{2\pi} \sqrt{4\sin^2(t/2)}\, dt \\ &= \int_0^{2\pi} 2\sin(t/2)\, dt \\ &= -4(\cos(\pi) - \cos(0)) = 8 \end{aligned}$$

Problem: The arc length of the helix $f(t) = \begin{bmatrix} a\cos t \\ a\sin t \\ bt \end{bmatrix}$, $0 \le t \le 2\pi$,

...is homework:)

Problem: The circumference of the elipse $\begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$, $a \neq b$

$$\int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt = 4a\mathrm{E}(e)$$

where $e = \sqrt{1 - (b/a)^2}$ is its *eccentricity* and the function E is the nonelementary *elliptic integral of 2nd kind*.