Mathematical modelling

Lecture 7, April 2, 2021

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The arc length from the initial $t = a$ to an arbitrary $t = T$

$$s(t) = \int_a^t \|f'(u)\| \, du$$

is an increasing function of t, so it has an inverse

$$t(s) : [0, s(T)] \to [a, T].$$

So, the original parameter t can be expressed as a funcion of the arc length s.

Inserting this into the parametrization gives the same curve with a different parametrization:

$$g(s) = f(t(s)).$$

The arc length is called the *natural parameter* of the curve.

Proposition

A curve C is parametrized with the natural parameter s satisfies

$$\|g'(s)\| = 1, \tag{1}$$

i.e., the length of the velocity vector is 1 at every point and so a parametrization with the natural parameter is the unit speed parametrization.

Proof. Indeed,

$$g'(s) = \frac{dg}{ds}(s) = \frac{d(f \circ t)}{ds}(s) = \frac{df}{dt}(t(s)) \cdot \frac{dt}{ds}(s) = f'(t(s))t'(s). \tag{2}$$

Now note that by the fundamental theorem of calculus we have that $s'(t) = \|f'(t)\|$ and hence $t'(s) = \frac{1}{\|f'(t(s))\|}$. Plugging this into (2) we get $g'(s) = \frac{f'(t(s))}{\|f'(t(s))\|}$, which is (1).

The natural parametrization of a curve is extremely important in theory, but for practical computing it is less useful.

Example

The standard parametrization of the circle

$$f(t) = \begin{bmatrix} a\cos t \\ a\sin t \end{bmatrix}$$

is not the natural parametrization if $a \neq 1$, since

$$\|f'(t)\| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = a \neq 1.$$

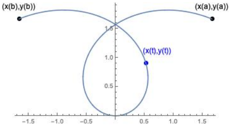
Since $s(t) = \int_0^t a\,dt = at$ it follows that $t = s/a$ and the natural parametrization is

$$g(s) = \begin{bmatrix} a\cos(s/a) \\ a\sin(s/a) \end{bmatrix}.$$

Remember:

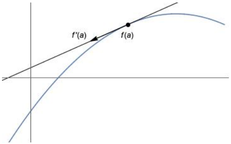
A *parametric curve*: $f(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix}$,

$t \in I \subset \mathbb{R}$,



The derivative $f'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_m'(t) \end{bmatrix}$

is the *velocity vector* or
tangent vector if $f'(t) \neq 0$,



The image $C = \{f(t), t \in I\}$: a (geometric) *curve* in $\mathbb{R}^m$.

A curve C has many parametrizations.

The *arc length* parametrization or *natural* parametrization $f(s)$:
s is the length of the chord from $f(a)$ to $f(s)$, $\|f'(s)\| = 1$.

Curvature

1. Intuitively we would like to measure for what amount does the curve deviate from being the straight line.
2. For the circle of radius R we would like that the curvature is proportional to $1/R$.

The *curvature* $\kappa(t)$ of a smooth curve $f(t)$ at a point $t = a$ is the rate of change of the unit tangent vector $T(t) = \dfrac{f'(t)}{\|f'(t)\|}$:

$$\kappa(t) = \left\| \frac{d}{dt} T(t) \right\|.$$

If the curve is parametrized by the arc length s, i.e., $\|f'(s)\| = 1$, then this is simply

$$\kappa(s) = \|f''(s)\|$$

Problem: what is the curvature of a circle with radius a?

The natural parametrization of the circle is $f(s) = \begin{bmatrix} a\cos(s/a) \\ a\sin(s/a) \end{bmatrix}$, so

$$f'(s) = \begin{bmatrix} -\sin(s/a) \\ \cos(s/a) \end{bmatrix} \qquad \text{and} \qquad f''(s) = \begin{bmatrix} -\cos(s/a)/a \\ -\sin(s/a)/a \end{bmatrix}.$$

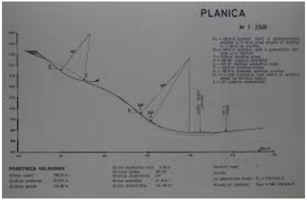
The curvature $\kappa(s) = \|f''(s)\| = 1/a$ is constant along the circle.

As $a \to \infty$, the circle goes towards a line and $\kappa \to 0$.

On the other hand, as $a \to 0$, the circle goes towards a point and $\kappa \to \infty$.

Problem: designing roads and railways

Roads, railway bends, roller coaster loops, the ski jump in Planica . . . are designed so that the transitions from the straight to the circular parts are as smooth as possible.

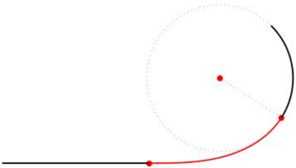


The force acting on a moving point on the curve (car, train, ski jumper,. . .) increases and decreses as evenly as possible.

The transition curve from

- the straight part (with curvature 0) to
- the circular part (with curvature $a > 0$)

has several names: *clotoid*, *Euler spiral*, *Cornu spiral* . . .



Its characteristic property is that the *curvature $\kappa(s)$ is a linear function of arc length s*.

Let us find its arc length parametrization $f(s)$. Assume that $\kappa(s) = \|f''(s)\| = 2s$.

Remember that the arc length parametrization is the unit speed parametrization, so $\|f'(s)\| = 1$ and so $f'(s)$ can be written in the form

$$f'(s) = \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix} = \begin{bmatrix} \cos\varphi(s) \\ \sin\varphi(s) \end{bmatrix}.$$

This gives

$$\kappa(s) = \sqrt{x''(s)^2 + y''(s)^2} = \varphi'(s) = 2s, \quad \varphi(s) = s^2,$$

$$x'(s) = \cos(s^2), \quad y'(s) = \sin(s^2),$$

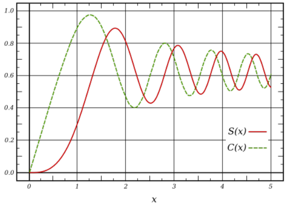
so

$$x(s) = \int_0^s \cos(u^2)\,du, \quad y(s) = \int_0^s \sin(u^2)\,du$$

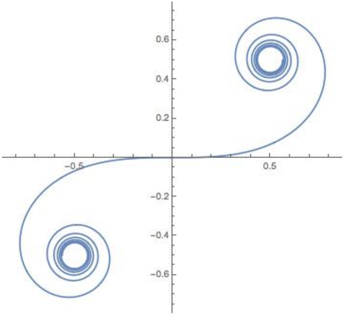
The functions

$$x(s) = \int_0^s \cos(u^2)\,du = C(s), \quad y(s) = \int_0^t \sin(u^2)\,du = S(s)$$

are nonelementary functions called the *Fresnel integrals*



Fresnel integrals



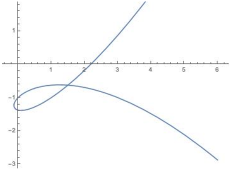
clotoid

3.3 Plane curves

Remember: for a plane curve $f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ the tangent at a regular point $f(a)$ is

- the vertical line $x = x(a)$ if $x'(a) = 0$ and $y'(a) \neq 0$,
- the line $y - y(a) = \dfrac{y'(a)}{x'(a)}(x - x(a))$ if $x'(a) \neq 0$,
- the horizontal line $y = y(a)$ if $y'(a) = 0$ and $x'(a) \neq 0$.

Plotting a parametric plane curve



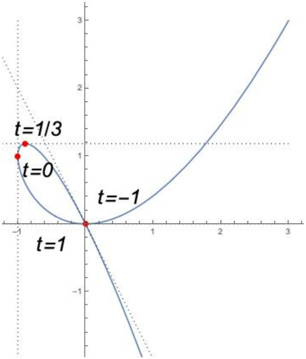
Here is a general strategy:

- find the asymptotic behaviour: $\lim_{t\to\infty} f(t)$, $\lim_{t\to-\infty} f(t)$
- find intersections with coordinate axes: solve $y(t) = 0$ and $x(t) = 0$
- find points where the tangent is vertical or horizontal: solve $x'(t) = 0$ and $y'(t) = 0$
- find self-intersections: solve $f(t) = f(s),\ t \neq s$
  - and the two tangents there
- look for other helpful features ...
- connect points $\mathbf{r}(t) = f(t)$ by increasing t

Problem: plot $f(t) = \begin{bmatrix} t^2 - 1 \\ -t^3 - t^2 + t + 1 \end{bmatrix}$

The derivative is $f'(t) = \begin{bmatrix} 2t \\ -3t^2 - 2t + 1 \end{bmatrix}$

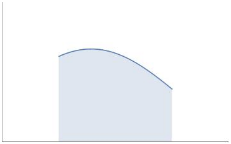
- Asymptotic behaviour: $\lim_{t \to \infty} f(t) = \begin{bmatrix} \infty \\ -\infty \end{bmatrix}$, $\lim_{t \to -\infty} f(t) = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$,
  - intersections with axes: $t = \pm 1$, at $(0, 0)$
    this is also a self-intersection
  - the two tangent lines at $(0, 0)$
    - at $t = -1$: $y = 0$,
    - at $t = 1$: $y = -2x$
  - vertical tangent: $t = 0$ at $(-1, 1)$
  - horizontal tangent
    - at $t_1 = -1$, $y = 0$,
    - at $t_2 = 1/3$, $y = 32/27$



Areas bounded by plane curves

I. Let $f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad t \in [a, b]$

$x'(t) > 0$



The area of the *quadrilateral bounded by the curve and the x-axis* is

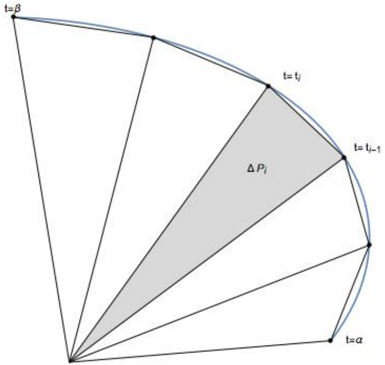
$$P = \int_{x(a)}^{x(b)} |y(x)|\, dx = \int_a^b |y(t)| x'(t)\, dt$$

Problem: the area under one arc of the cycloid:

$$x(t) = at - a\sin t, \quad y(t) = a - a\cos t,$$

$$P = \int_0^{2\pi} a^2 (1 - \cos t)^2\, dt = a^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos(2t) \right) dt = 3a^2\pi.$$

II. The area of the *triangular region bounded by the curve* $f(t), t \in [a, b]$, *and the two end-point position vectors* $f(a)$ and $f(b)$:



An approximate value of the area is the sum of areas of triangles obtained by subdividing the interval $[a, b]$ into n intervals of length $\Delta t = (b - a)/n$.

The area of a triangle with vertices $(0,0)$, $f(t_i)$, $f(t_{i+1})$ is

$$\Delta P_i = \frac{1}{2}\|f(t_{i+1}) \times f(t_i)\| \doteq \frac{1}{2}\|(f(t_i) + f'(t_i)\Delta t) \times f(t_i)\|$$

$$= \frac{1}{2}\|f'(t_i) \times f(t_i)\|\Delta t = \frac{1}{2}|y'(t_i)x(t_i) - x'(t_i)y(t_i)|\Delta t.$$

The area is obtained by adding these and letting $n \to \infty$:

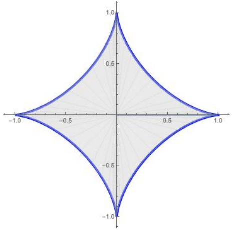
$$P = \lim_{n\to\infty} \frac{1}{2} \sum_{i=0}^{n-1} |y'(t_i)x(t_i) - x'(t_i)y(t_i)|\Delta t$$

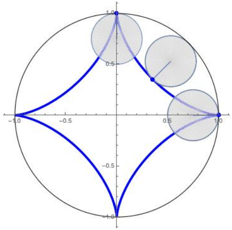
$$P = \frac{1}{2}\int_a^b |x(t)y'(t) - y(t)x'(t)|\, dt.$$

Problem: the area bounded by

1. the asteroid $x(t) = \cos^3 t, y(t) = \sin^3 t, t \in [0, 2\pi]$ is

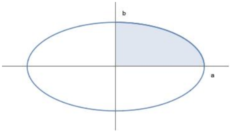
$$P = \cdots = \frac{3\pi}{8}$$





2. the elipse $x = a\cos t,\ y = b\sin t, t \in [0, 2\pi]$ is

$$P = \frac{4}{2}\int_0^{\pi/2} ab(\cos^2 t + \sin^2 t) = ab\pi.$$



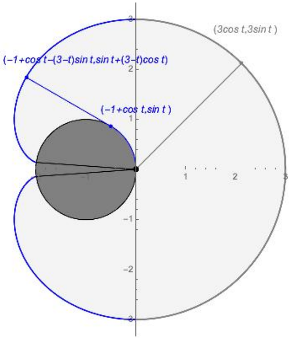
Problem: a circular tower with radius 1 is in the middle of a grassy lawn. A goat is tied to the tower with a rope of length 3 meters. What is the total area of grass that the goat can reach?

Center of the tower: $(-1, 0)$,
goat tied to the tower at $(0, 0)$
area of grass bounded by:

$$f(t) = \begin{bmatrix} 3\cos t \\ 3\sin t \end{bmatrix}, t \in [-\pi/2, \pi.2]$$

$$g_1(t) = \begin{bmatrix} -1 + \cos t - (3 - t)\sin t \\ \sin t + (3 - t)\cos t \end{bmatrix}, t \in [0, 3],$$

$$g_2(t) = \begin{bmatrix} -1 + \cos t + (3 + t)\sin t \\ \sin t - (3 - t)\cos t \end{bmatrix}, t \in [-3, 0]$$



$$A = 9\pi/2 + 2\int_0^3 |x_1(t)y_1'(t) - y_1(t)x_1'(t)|dt = 14 - 2\sin(3) + 9\pi/2.$$