Mathematical modelling

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4. Differential equations and dynamic models

Definition of an ordinary differential equation

Ordinary differential equation, ODE, is an equation of an unknown function and an independent variable. ODE relates the independent variable with the function and its derivatives.

If t is an independent variable, $x(t)$ is a function of t, then the ODE is of the form:

$$F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0.$$

Similarly if x is an independent variable, $y(x)$ a function of x, then the ODE is of the form:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

The *order* of a differential equation is the order of the highest derivative.

- $\dot{x} - 3t^2 = 0$.

  So,
  $$\frac{dx}{dt} = 3t^2 \;\Rightarrow\; x(t) = t^3 + C, \quad \text{where } C \text{ is a constant.}$$
  If we want to determine C, we need an additional condition, e.g., *initial condition* $x(0) = x_0$, $x_0 \in \mathbb{R}$, or any other condition $x(t_0) = x_0$, $x_0 \in \mathbb{R}$.

- $y''(x) + 2y'(x) = 3y(x)$.

  We will learn how to solve such an ODE, but right now let us only check that $y(x) = Ce^{-3x}$, $C \in \mathbb{R}$ a constant, is a solution:
  - Calculate $y''(x)$, $y'(x)$:
    $$y'(x) = -3Ce^{-3x}, \quad y''(x) = 9Ce^{-3x}.$$
  - Plug into the given ODE:
    $$9Ce^{-3x} - 6Ce^{-3x} = 3Ce^{-3x}.$$

- $\cos t \cdot \ddot{x} - 3t^4 \cdot \dot{x} + 5e^t = 0.$

  Such ODE's cannot be solved analytically (or are at least hard to solve). We will learn how to solve such ODE's by using numerical methods.

Partial differential equation, PDE, is an equation for an unknown function u of $n \geq 2$ independent variables, e.g., for $n = 2$ we have

$$F(x, y, u_x, u_y, u_{xx}, \dots) = 0,$$

where x, y are the independent variables.

We will not consider PDE's, from now on DE means an ODE.

Differential equations are used for modelling a *deterministic process*: a law relating a certain quantity depending on some independent variable (for example time) with its rate of change, and higher derivatives.

1. *Newton's law of cooling:*

$$\dot{T} = k(T - T_\infty), \tag{1}$$

where $T(t)$ is the temperature of a homogeneous body (can of beer) at time t, T_0 is the initial temperature at time $t_0 = 0$, T_∞ is the temperature of the environment, k is a constant (heat transfer coefficient).

(1) is an example of a separable ODE and also the first order linear ODE. We will see shortly how to solve such types of ODE's. For now you can check easily by yourself that the solution is

$$T(t) = (T_0 - T_\infty)e^{kt}.$$

2. *Radioactive decay:*

$$\dot{y}(t) = -ky(t), \quad k = \frac{\log 2}{t_{1/2}},$$

where $y(t)$ is the remaining quantity of a radioactive isotope at time t, $t_{1/2}$ is the *half-life* and k is the *decay constant*. The solution is

$$y(t) = Ce^{-kt}, \quad \text{where } C \text{ is a constant.}$$

Let's verify, that $t_{1/2}$ really represents the time in which the amount of the isotope decreases to half of its current amount. At time $t = 0$ the amount is $y(0) = Ce^0 = C$. We have to check that $y(t_{1/2}) = \frac{C}{2}$:

$$y(t_{1/2}) = Ce^{-\frac{k \log 2}{k}} = Ce^{-\log 2} = Ce^{\log 1/2} = \frac{C}{2}.$$

3. Simple harmonic *oscillator*:

$$\ddot{x} + \omega x = 0.$$

The function $x(t)$ is a *solution* of a DE

$$F(t, x, \dot{x}, \ddot{x}, \dots, x^{(n)}) = 0$$

on an interval I if it is at least n times differentiable and satisfies the identity

$$F(t, x(t), \dot{x}(t), \ddot{x}(t), \dots, x^{(n)}(t)) = 0$$

for all $t \in I$.

Analytically solving a differential equation is typically very difficult, very often impossible.

To find approximate solutions we use different simplifications and numerical methods.

4.1. First order ODE's

Definition of a first order DE

We will (mostly) consider first order ODE's in the form

$$\dot{x} = f(t, x).$$

- The *general solution* is a one-parametric family of solutions $x = x(t, C)$.
- A *particular solution* is a specific function from the general solution, that usually satisfies some *initial condition* $x(t_0) = x_0$.
- A *singular solution* is an exceptional solution that is not part of the general solution.

We will first look at some simple types of 1.-st order DE's that are analytically solvable.

A *separable* DE is of the form

$$\dot{x} = f(t)g(x). \tag{2}$$

This can be solved by:

- Inserting $\dot{x} = \dfrac{dx}{dt}$ into (2):

$$\frac{dx}{dt} = f(t)g(x). \tag{3}$$

- *Separating variables* in (3):

$$\frac{dx}{g(x)} = f(t)\,dt. \tag{4}$$

- Integrating both sides of (3):

$$\int \frac{1}{g(x)}\,dx = \int f(t)\,dt + C$$

Example 1 of a separable DE

$$\boxed{\dot{x} = kx} \quad \text{where } k \in \mathbb{R} \text{ is a fixed real number} \tag{5}$$

- $$\frac{dx}{dt} = kx,$$
- $$\frac{dx}{x} = kdt,$$
- $$\log|x| = \int \frac{dx}{x} = \int k\,dt = kt + C,$$

where C is a constant and so

$$|x| = e^{kt+C}$$

is a general solution to (5). Clearly, $x(t) = 0$ is also a solution of the equation. By introducing a new constant e^C which, by abuse of notation, we again denote by C, this is equivalent to

$$x(t) = Ce^{kt}, C \in \mathbb{R}.$$

Example 2 of a separable DE

$$\boxed{\dot{x} = kx(1-x)} \quad \text{where } k \in \mathbb{R} \text{ is a fixed real number} \tag{6}$$

- $$\frac{dx}{dt} = kx(1-x),$$
- $$\frac{dx}{x(1-x)} = kdt,$$
- By the method of partial fractions we get

$$\log\left|\frac{x}{1-x}\right| = \log|x| - \log|1-x| = \int \frac{dx}{x} - \int \frac{dx}{1-x} = \int k\,dt = kt + C,$$

where C is a constant and so

$$\frac{x}{1-x} = Ce^{kt}.$$

Expressing $x(t)$ we get

$$x(t) = \frac{1}{Ce^{-kt} + 1} \tag{7}$$

is a general solution to (6). $x(t)$ from (7) is called a *logistic function*.

Example 3 of a separable DE

$$\boxed{y' = \frac{-x}{ye^{x^2}},} \quad y(0) = 1. \tag{8}$$

- $$\frac{dy}{dx} = \frac{-x}{ye^{x^2}},$$
- $$ydy = -xe^{-x^2}dx,$$
- Integrating:
$$\frac{y^2}{2} = \int ydy = \int (-xe^{-x^2})dx = \frac{1}{2}e^{-x^2} + C,$$
where C is a constant.
- $\frac{1}{2} = \frac{y^2(0)}{2} = \frac{1}{2} + C \;\Rightarrow\; C = 0.$

Expressing $y(x)$ we get $y(x) = \pm\sqrt{e^{-x^2}}$ and since $y(0) > 0$ we have
$$y(x) = \sqrt{e^{-x^2}}.$$

Let $x(t)$ be the size of a population (bacteria, trees, people, ...) at time t. The most common models for population growth are:

- **exponential growth**: the growth rate is proportional to the size, modelled by $\dot{x} = kx$, with the solution the exponential function $x(t) = x_0 e^{kt}$, where $x_0 = x(0)$ is the initial population size.
- **logistic growth**: the growth rate is proportional to the size and the resources, modelled by $\dot{x} = kx(1 - x/x_{max})$, where x_{max} is the capacity of the environment, i.e., maximal population size that it still supports, with the solution is the logistic function.
- **general model**: the growth rate is proportional to the size, but the proportionality factor depends on time and size, modelled by $\dot{x} = k(x,t)f(x)$; the equation is not separable and is analytically solvable only in very specific cases.

$x(t)$ is the ratio of people in a given group that at time t knows a certain piece of information.

Let $x_0 = x(t_0)$ be the 'informed' ratio at time $t = t_0$.

Consider two possible models:

- spreading through an external source: the rate of change is proportional to the uninformed ratio $\dot{x} = k(1 - x)$ with $x_0 = 0$,
- spreading through "word of mouth" the rate of change is proportional to the number of encounters between informed and uninformed members $\dot{x} = kx(1 - x)$ *logistic law, again*, with $x_0 > 0$.

A *first order linear DE* is of the form

$$\dot{x} + f(t)x = g(t) \tag{9}$$

The equation is *homogeneous* if $g(t) = 0$ and *nonhomogenous* if $g(t) \neq 0$.

A homogeneous part of (9),

$$\dot{x} + f(t)x = 0, \tag{10}$$

has a general solution of the form

$$Cx_h(t), \tag{11}$$

where $C \in \mathbb{R}$ is a constant and $x_h(t)$ is a particular solution. Indeed:

- Every $x(t)$ of the form (11) is a solution of (10):

$$\begin{aligned} x'(t) + f(t)x(t) &= (Cx_h)'(t) + f(t)Cx_h(t) \\ &= Cx_h'(t) + f(t)Cx_h(t) \\ &= C(x_h'(t) + f(t)x_h(t)) \\ &= 0 \end{aligned}$$

- If $x(t)$ is a solution of (10), then it must be of the form (11). Indeed, since $x(t)$ and $x_h(t)$ both solve (10),

$$\begin{aligned}\left(\frac{x(t)}{x_h(t)}\right)' &= \frac{x'(t)x_h(t) - x(t)x_h'(t)}{x_h^2(t)} \\ &= \frac{-f(t)x(t)x_h(t) + f(t)x(t)x_h(t)}{x_h^2(t)} \\ &= 0.\end{aligned}$$

Hence, $\frac{x(t)}{x_h(t)} = C$ for some constant C and $x(t)$ is of the form (11).

Let $x_p(t)$ be any particular solution of (9):

$$g(t) = x_p'(t) + f(t)x_p(t) \tag{12}$$

The general solution of (9) is a sum

$$x(t) = Cx_h(t) + x_p(t). \tag{13}$$

Indeed:

- Every $x(t)$ of the form (13) is a solution of (9):

$$\begin{aligned} x'(t) + f(t)x(t) &= (Cx_h(t) + x_p(t))' + f(t)(Cx_h(t) + x_p(t)) \\ &= Cx_h'(t) + x_p'(t) + f(t)Cx_h(t) + f(t)x_p(t) \\ &= (Cx_h'(t) + f(t)Cx_h(t)) + (x_p'(t) + f(t)x_p(t)) \\ &= 0 + g(t), \end{aligned}$$

  where we used (12) in the last equality.
- If $x(t)$ is a solution of (9), then it must be of the form (13). Indeed, since $x(t)$ and $x_p(t)$ both solve (9), $x(t) - x_p(t)$ solves the homogenous part (10) of (9). Hence, $x(t) - x_p(t) = Cx_h(t)$ for some C and $x(t) = Cx_h(t) + x_p(t)$.

The particular solution x_p can be obtained by *variation of the constant*, that is, by substituting the constant C is the homogenous solution by an unknown function $C(t)$ which is then determined from the equation.

$$\boxed{t^2\dot{x} + tx = 1}, \quad \boxed{x(1) = 2}. \tag{14}$$

1. The homogenous part is

$$t^2\dot{x} + tx = 0. \tag{15}$$

So the solution x_h to (15) is

$$t^2\,dx = -tx\;dt \;\Rightarrow\; \frac{dx}{x} = -\frac{dt}{t} \;\Rightarrow\; \log|x| = -\log|t| + \log C = \log\frac{C}{|t|}$$

$$\Rightarrow\; x_h = \frac{C}{t}.$$

2. A particular solution of the nonhomogenous equation is obtained by variation of the constant:

$$x = \frac{C(t)}{t}, \quad \dot{x} = \frac{C'(t)t - C(t)}{t^2}$$

by inserting into (14) we obtain

$$C'(t)t - C(t) + C(t) = 1 \;\Rightarrow\; C'(t) = \frac{1}{t} \;\Rightarrow\; C(t) = \log|t|.$$

3. So the general solution of the nonhomogenous equation is

$$x(t) = \frac{C}{t} + \frac{\log|t|}{t}. \tag{16}$$

4. Finally, since $x(1) = 2$, we get by plugging $t = 1$ into (16)

$$2 = x(1) = C$$

and hence the solution of (14) is

$$x(t) = \frac{2 + \log|t|}{t}.$$

$$\boxed{y'(x) = f(x)y(x) + g(x)}. \tag{17}$$

1. The homogenous part is

$$y'(x) = f(x)y(x). \tag{18}$$

So the solution $y(x)$ to (18) is

$$\log |y| = \int \frac{dy}{y} = \int f(x)dx + C \;\Rightarrow\; y(x) = C \cdot e^{\int f(x)dx}$$

2. A particular solution of the nonhomogenous equation is obtained by the variation of the constant:

$$y(x) = C(x) \cdot e^{\int f(x)dx}$$

$$y'(x) = C'(x) \cdot e^{\int f(x)dx} + C(x)f(x)e^{\int f(x)dx}$$

By inserting into (17) we obtain

$$C'(x) \cdot e^{\int f(x)dx} + C(x)f(x)e^{\int f(x)dx} = f(x)C(x) \cdot e^{\int f(x)dx} + g(x)$$

Hence

$$C'(x) \cdot e^{\int f(x)dx} = g(x),$$

and so

$$C(x) = \int (g(x)e^{-\int f(x)dx})dx.$$

Finally the solution is

$$y(x) = e^{\int f(x)dx}(C + \int (g(x)e^{-\int f(x)dx})dx).$$

In the example $t^2\dot{x} + tx = 1$ (or $\dot{x} = -\frac{1}{t}x + \frac{1}{t^2}$) above we get

$$\begin{aligned} x(t) &= e^{\int -\frac{1}{t}dt}(C + \int (\frac{1}{t^2}e^{\int \frac{1}{t}dt})dt) \\ &= e^{\log|\frac{1}{t}|}(C + \int (\frac{1}{t^2}t)dt) \\ &= \frac{1}{t}(C + \log|t|). \end{aligned}$$

A ball of mass m kg is thrown vertically into the air with initial velocity $v_0 = 10$ m/s. We follow its trajectory. By Newton's second law of motion,

$$F = ma,$$

where m is the mass, $a = \dot{v} = \ddot{x}$ is acceleration and v velocity, and F is the sum of forces acting on the ball.

- Assuming no air friction the model is

  $$m\dot{v} = -mg,$$

  where g is the gravitational constant. The solution is

  $$v = -gt + C \quad \text{where } C \text{ is a constant.}$$

- Assuming the *linear law of resistance (drag)* $F_u = -kv$ the model is

  $$m\dot{v} = -mg - kv.$$

  The solution is $v = v_h + v_p$ where

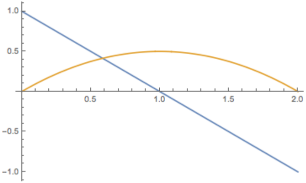
  $$v_h = Ce^{-kt/m} \quad \text{and} \quad v_p = -mg/k.$$

Motion of ball in the case $m = 1$, $k = 1$ and approximating $g \doteq 10$ (we will omit units)

Model | Velocity and position | Solution

$ma = -mg$

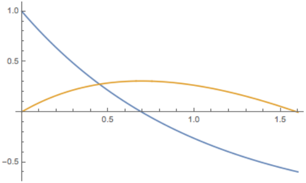
$\dot{v} = -10$



$v(t) = -10t + 10$
$x(t) = -5t^2 + 10t$

$ma = -mg - kv$

$\dot{v} = -v - 10$



$v(t) = 20e^{-t} - 10$
$x(t) = 20 - 20e^{-t} - 10t$

The ball reaches the top at time t where $v(t) = 0$ and the ground at time t where $x(t) = 0$.

- Assuming no friction, the ball is at the top at $t = 10$.

  At time $t = 1$, $x(t) = 0$, so it takes the same time going up and falling down.

- Assuming linear friction, the ball reaches the top at $t = \log 2$.

  At time $2 \log 2$, $x(2 \log 2) = 20 - 5 - 20 \log 2 > 0$ so it takes longer falling down than going up.

A *homogeneous* (nonlinear) DE is of the form

$$\dot{x} = f\left(\frac{x}{t}\right). \tag{19}$$

The solution is obtained by introducing a new dependent variable

$$u = \frac{x}{t}.$$

Hence $x = ut$ and differentiating with respect to t we get

$$\dot{x} = \dot{u}t + u. \tag{20}$$

Plugging (20) into (19) we get

$$\dot{u}t + u = f(u). \tag{21}$$

Rearranging (21) we obtain

$$t\dot{u} = f(u) - u,$$

which is a separable DE.

$$\boxed{y' = \frac{y-x}{x}}$$

can be written as

$$y' = \frac{y}{x} - 1. \tag{22}$$

Introducing a new dependent variable

$$u = \frac{y}{x},$$

plugging in (22), we get

$$u'x + u = u - 1. \tag{23}$$

This is equivalent to

$$u'x = -1$$

and hence

$$u = \frac{y}{x} = \log(\frac{C}{x}).$$

Orthogonal trajectories

Definition and the procedure for solving

Given a 1-parametric family of curves

$$F(x, y, a) = 0 \quad \text{where} \quad a \in \mathbb{R},$$

an *orthogonal trajectory* is a curve

$$G(x, y) = 0$$

that intersects each curve from the given family at a right angle.
Assume that F is a differentiable function.

1. The family $F(x, y, a) = 0$ is the general solution of a 1st order DE, that is obtained by differentiating the equation with respect to the independent variable (using implicit differentiation) and eliminating the parameter a.
2. By substituting y' for $-1/y'$ in the DE for the original family, we obtain a DE for curves with orthogonal tangents at every point of intersection.
3. The general solution to this equation is the family of orthogonal trajectories to the original equation.

Example - orthogonal trajectories to the family of circles

Let us find the orthogonal trajectories to the family of circles through the origin with centers on the y axis:

$$x^2 + y^2 - 2ay = 0. \tag{24}$$

Differentiating (24) w.r.t. the independent variable gives

$$2x + 2yy' - 2ay' = 0. \tag{25}$$

Expressing a from (25) gives

$$a = \frac{x}{y'} + y. \tag{26}$$

Inserting (26) into (24) we obtain the DE for the given family

$$x^2 - y^2 - \frac{2xy}{y'} = 0. \tag{27}$$

Next we express y' from (27) and obtain

$$y' = \frac{2xy}{x^2 - y^2}. \tag{28}$$

The DE for orthogonal trajectories is obtained by substituting y' for $-1/y'$ in (28) to obtain

$$-\frac{1}{y'} = \frac{2xy}{x^2 - y^2}, \tag{29}$$

which is equivalent to

$$y' = -\frac{x^2 - y^2}{2xy}. \tag{30}$$

(30) is a homogeneous DE:

$$y' = -\frac{x^2 - y^2}{2xy} = -\frac{x}{2y} + \frac{y}{2x}$$

By introducing $y = ux$ we obtain

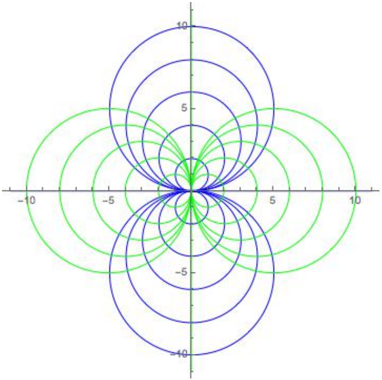
$$\begin{aligned} u'x + u = -\frac{1}{2u} + \frac{u}{2} \quad &\Rightarrow \quad u'x = -\frac{1+u^2}{2u} \quad \Rightarrow \quad \frac{2udu}{1+u^2} = -\frac{dx}{x} \\ &\Rightarrow \quad \log(1+u^2) = -\log x + \log C \\ &\Rightarrow \quad 1 + u^2 = \frac{C}{x}, \end{aligned}$$

Plugging in $u = \frac{y}{x}$ again gives the general solution

$$x^2 + y^2 = Cx.$$

Orthogonal trajectories to circles through the origin with centers on the y axis are circles through the origin with centers on the x axis.

Both families together form an orthogonal net:



Let $D \subset \mathbb{R}^2$ be the domain of the function $f(x, y)$. For each point $(x, y) \in D$ the DE

$$y' = f(x, y)$$

gives the value y' of the coefficient of the tangent to the solution $y(x)$ through this specific point, that is, the direction in which the solution passes through the point.

All these directions together form the *directional field* of the equation.

A solution of the equation is represented by a curve $y = y(x)$ that follows the given directions at every point x, i.e., the coefficient of the tangent corresponds to the value $f(x, y(x))$.

The general solution to the equation is a family of curves, such that each of them follows the given direction.

Directional fields and solutions of

$$y' = ky \qquad\qquad y' = ky(1-y)$$