Mathematical modelling

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Notice first that a 1st order DE

$$\dot{x} = f(t, x)$$

can be rewritten in the form

$$M(t,x)dt + N(t,x)dx = 0. \tag{1}$$

Recall that the differential of a function $u(t,x)$ is equal to

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) \cdot (dt, dx),$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^2$.
DE (1) is *exact* if there exists a differentiable function $u(t,x)$ such that

$$\frac{\partial u}{\partial t} = M(t,x) \quad \text{and} \quad \frac{\partial u}{\partial x} = N(t,x).$$

In this case, solutions of (1) are level curves of the function u:

$$u(t,x) = C, \qquad \text{where} \quad C \in \mathbb{R}.$$

Recall from Calculus that if u has continuous second order partial derivatives then

$$\frac{\partial u}{\partial x \partial t} = \frac{\partial u}{\partial t \partial x}.$$

This gives the following necessary condition for exact differential equations

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}. \tag{2}$$

If M and N are differentiable for every $(t, x) \in \mathbb{R}^2$, the condition (2) is also sufficient.

A *potential function* u can be determined from the following equality

$$u(x, t) = \int M(t, x)\, dt + C(x) = \int N(t, x)\, dx + D(t),$$

where $C(x)$ and $D(t)$ are some functions.

The DE

$$x + ye^{2xy} + xe^{2xy}y' = 0$$

can be rewritten as

$$(x + ye^{2xy})dx + xe^{2xy}dy = 0.$$

The equation is exact since

$$\frac{\partial(x + ye^{2xy})}{\partial y} = \frac{\partial(xe^{2xy})}{\partial x} = (e^{2xy} + 2xye^{2xy}).$$

A potential function is equal to

$$\begin{aligned} u(x, y) &= \int (x + ye^{2xy})\, dx = \frac{x^2}{2} + \frac{1}{2}e^{2xy} + C(y) \\ &= \int (xe^{2xy})\, dy = \frac{1}{2}e^{2xy} + D(x), \end{aligned}$$

Defining $C(y) = 0$ and $D(x) = x^2/2$, we get $u(x, y) = \frac{x^2}{2} + \frac{1}{2}e^{2xy}$. The general solution is the family of level curves

$$\frac{x^2}{2} + \frac{1}{2}e^{2xy} = E, \quad E \in \mathbb{R}.$$

Theorem (Existence and uniqueness of solutions)

If $f(x, y)$ is continuous and differentiable with respect to y on the rectangle

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b], \quad a, b > 0$$

then the DE with initial condition

$$y' = f(x, y), \quad y(x_0) = y_0,$$

has a unique solution $y(x)$ defined at least on the interval

$$[x_0 - \alpha, x_0 + \alpha], \quad \alpha = \min\left\{a, \frac{b}{M}, \frac{1}{N}\right\},$$

where

$$M = \max\{f(x, y) \colon (x, y) \in D\} \text{ and } N = \max\left\{\frac{\partial f(x, y)}{\partial y} \colon (x, y) \in D\right\}.$$

We are given the DE with the initial condition

$$y'(x) = f(y, x), \quad y(x_0) = y_0.$$

Instead of analytically finding the solution $y(x)$, we construct a recursive sequence of points

$$x_i = x_0 + ih, \quad y_i \doteq y(x_i), i \geq 0$$

where y_i is an approximation to the value of the exact solution $y(x_i)$, and h is the *step size*.

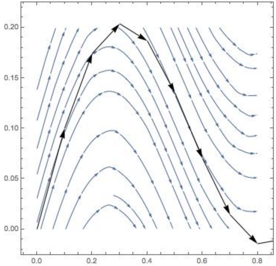
A number of numerical methods exists, the choice depends on the type of equation, desired accuracy, computational time,...

We will look at the simplest and best known.

Euler's method is the simplest and most intuitive approach to numerically solve a DE.

At each step the value y_{i+1} is obtained as the point on the tangent to the solution through (x_i, y_i) at $x_{i+1} = x_i + h$:

- initial condition: (x_0, y_0)
- for each i: $x_{i+1} = x_i + h$, $y_{i+1} = y_i + hf(x_i, y_i)$.



The point (x_{i+1}, y_{i+1}) typically lies on a different particular solution than (x_i, y_i), at each step, the error at each step is of order $\mathcal{O}(h^2)$. The cumulative error is of order $\mathcal{O}(h)$.

The *Runge-Kutta* method is the most widely used numerical method for DE's:

- initial condition: (x_0, y_0)
- at each step x_i is computed as a weighted average of approximations at $x = x_i$, $x = x_i + h/2$ and $x = x_{i+1}$:

$$x_{i+1} = x_i + h, \quad y_{i+1} = y_i + (k_1 + 2k_2 + 2k_3 + k_4)/6,$$

  - $k_1 = h \cdot f(x_i, y_i)$,
  - $k_2 = h \cdot f(x_i + h/2, y_i + k_1/2)$,
  - $k_3 = h \cdot f(x_i + h/2, y_i + k_2/2)$ in
  - $k_4 = h \cdot f(x_i + h, y_i + k_3)$

The error at each step is of order $\mathcal{O}(h^5)$. The cumulative error is of order $\mathcal{O}(h^4)$.

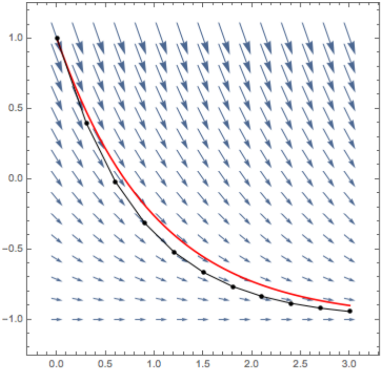
Euler vs Runge-Kutta

Below is a comparison of Euler's and Runge-Kutta methods for the DE

$$y' = -y - 1, \quad y(0) = 1 \quad \text{with step size } h = 0.3:$$

The red curve is the exact solution $y = 2e^{-x} - 1$.

Euler's method



Runge-Kutta