Mathematical modelling

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4.2. Systems of first order ODE's

Formal definition

Let

$$f := (f_1, \dots, f_n) : \mathbb{R}^{n+1} \to \mathbb{R}^n,$$

$$f(x_1, \dots, x_{n+1}) = (f_1(x_1, \dots, x_{n+1}), \dots, f_n(x_1, \dots, x_{n+1})).$$

be a vector function. A *system* of first order DE's is an equation

$$\dot{x}(t) = f(x(t), t), \tag{1}$$

where

$$x(t) := (x_1(t), \dots, x_n(t)) : I \to \mathbb{R}^n$$

is an unknown vector function and $I \subset \mathbb{R}$ is some interval. Coordinate-wise the system (1) is equal to

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), \dots, x_n(t), t), \\ &\vdots \\ \dot{x}_n(t) &= f_n(x_1(t), \dots, x_n(t), t). \end{aligned}$$

For every $(x, t) \in \mathbb{R}^{n+1}$ in the domain of f, the value $f(x, t)$ is the tangent vector $\dot{x}(t)$ to the solution $x(t)$ at the given t.

The *general solution* is a family of parametric curves

$$x(t, C_1, \ldots, C_n),$$

where $C_1, C_2, \ldots, C_n \in \mathbb{R}$ are parameters, with the given tangent vectors.

An *initial condition*

$$x(t_0) = x_0 \in \mathbb{R}^n$$

gives a *particular solution*, that is, a specific parametric curve from the general solution that goes through the point x_0 at time t_0.

A linear system of DEs is of the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}, \qquad (2)$$

where

$$x_i : I \to \mathbb{R}, \quad a_{ij} : I \to \mathbb{R} \quad \text{and} \quad g_i : I \to \mathbb{R}$$

are functions of t and $I \subseteq \mathbb{R}$ is an interval. In a compact form (2) can be written as

$$\dot{x}(t) = A(t)x + g(t), \qquad (3)$$

where

$$A(t) = [a_{ij}(t)]_{i,j=1}^{n}$$

is a $n \times n$ matricial function and

$$g(t) = \begin{bmatrix} g_1(t) & \dots & g_n(t) \end{bmatrix}^T$$

is a $n \times 1$ vector function.

The system (3)

- is *homogeneous* if for every t in the domain I we have $g(t) = \mathbf{0}$.
- has *constant coefficients*, if the matrix A is constant, i.e., independent of t.
- is *autonomous*, if it is homogeneous and has constant coefficients.

An autonomous linear system

$$\dot{x} = Ax \tag{4}$$

of 1st order DEs can be solved analytically, using methods from linear algebra. Recall that such a system can be written in coordinates as:

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,\\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n,\\
&\vdots\\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n.
\end{aligned}$$

Assume first that the matrix *A in (4) is diagonal*. Then (4) is the following:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Or equivalently,

$$\dot{x}_1 = \lambda_1 x_1, \qquad \dot{x}_2 = \lambda_2 x_2, \qquad \dots \qquad , \qquad \dot{x}_n = \lambda_n x_n.$$

In this (simple) case the general solution is easily determined:

$$x(t) = \begin{bmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{bmatrix} = C_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + C_2 e^{\lambda_2 t} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + C_n e^{\lambda_n t} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Assume next, that *A in (4) has n linearly independent eigenvectors* $v_1, \dots v_n$ with the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$.

- For every fixed t, the vector $x(t)$ can be expressed as a linear combination
$$x(t) = \varphi_1(t)v_1 + \cdots + \varphi_n(t)v_n.$$
- Hence, the coefficients
$$\varphi_i(t) : I \to \mathbb{R}, \qquad i = 1, \dots, n,$$
are functions of t.
- Since $v_1, \dots v_n$ are eigenvectors it follows from $\dot{x} = Ax$, that
$$\sum_{i=1}^{n} \dot{\varphi}_i(t)v_i = \sum_{i=1}^{n} \varphi_i(t)Av_i = \sum_{i=1}^{n} \varphi_i(t)\lambda_i v_i.$$
- Since $v_1, \dots v_n$ are linearly independent, it follows that for every i we have
$$\dot{\varphi}_i(t) = \lambda_i \varphi_i(t) \quad \Rightarrow \quad \varphi_i(t) = C_i e^{\lambda_i t}, \quad C_i \in \mathbb{R}.$$
- Hence the general solution of the system is
$$x(t) = C_1 e^{\lambda_1 t} v_1 + \cdots + C_n e^{\lambda_n t} v_n.$$

Find the general solution of the system

$$\dot{x}_1 = x_1 + x_2,$$
$$\dot{x}_2 = 4x_1 - 2x_2.$$

The matrix of the system is

$$A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}.$$

Its eigenvalues are the solutions of

$$\det(A - \lambda I) = (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 6 = 0,$$

so $\lambda_1 = -3$ and $\lambda_2 = 2$, and the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The general solution of the system is

$$x(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find the general solution of

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -4x_1.$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.$$

It has a conjugate pair of complex eigenvalues and a corresponding conjugate pair of eigenvectors:

$$\lambda_{1,2} = \pm 2i, \quad v_{1,2} = \begin{bmatrix} 1 \\ \pm 2i \end{bmatrix}.$$

The general solution is a family of complex valued functions

$$x(t) = C_1 e^{2it} \begin{bmatrix} 1 \\ 2i \end{bmatrix} + C_2 e^{-2it} \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

(which is not very useful in modelling real-valued phenomena).

Assume that the matrix of the system *A has a complex pair of eigenvalues* $\lambda_{1,2} = \alpha \pm i\beta$ and corresponding eigenvectors $v_{1,2} = u \pm iw$.

The real and imaginary parts of the two complex valued solutions are:

$$\begin{aligned} & e^{(\alpha \pm i\beta)t}(u \pm iw) \\ = \; & e^{\alpha t}(\cos(\beta t) \pm i \sin(\beta t))(u \pm iw) \\ = \; & e^{\alpha t}\left[\cos(\beta t)u - \sin(\beta t)w \pm i(\sin(\beta t)u + \cos(\beta t)w)\right]. \end{aligned}$$

Any linear combination (with coefficients $C_1, C_2 \in \mathbb{R}$) of these is a real-valued solution, so the real-valued general solution is

$$x(t) = e^{\alpha t}\left[C_1(\cos(\beta t)u - \sin(\beta t)w) + C_2(\sin(\beta t)u + \cos(\beta t)w)\right].$$

In the case of Example 2, $\lambda_{1,2} = \pm 2i$, i.e. $\alpha = 0$ and $\beta = 2$, and

$$v_{1,2} = \begin{bmatrix} 1 \\ \pm 2i \end{bmatrix} \quad \Rightarrow \quad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Hence, the general solution is

$$\begin{aligned} x(t) = C_1 \Big(\cos(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Big) \\ + C_2 \Big(\sin(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(2t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Big). \end{aligned}$$

If A has *less than n linearly independent eigenvectors*, additional solutions can also be obtained (e.g., with the use of Jordan form of A), but we will not consider this case here.

The *general solution* of a system $\dot{x} = Ax$ of n equations is of the form

$$x(t) = C_1 x^{(1)}(t) + \ldots + C_n x^{(n)}(t),$$

where $x^{(1)}(t), \ldots, x^{(n)}(t)$ are specific, linearly independent solutions.

For every eigenvalue $\lambda \in \mathbb{R}$ or a pair of eigenvalues $\lambda = \alpha \pm i\beta$ we obtain as many solutions as there are corresponding linearly independent eigenvectors.

An *initial condition* $x(t_0) = x^{(0)}$ gives a nonsingular system (if the vectors $x_1(t_0), \ldots, x_n(t_0)$ are linearly independent) of n linear equations for the constants $C_1, \ldots, C_n$.

$$x^{(0)} = C_1 x_1(t_0) + \ldots + C_n x_n(t_0).$$

This implies that a problem

$$\dot{x} = Ax, \quad x(t_0) = x^{(0)}$$

has a unique solution for any $x^{(0)}$.

Example

The initial condition $x^{(0)} = x(0) = \begin{bmatrix} 0 & 5 \end{bmatrix}^T$ for the system in Example 1, gives the following system of equations for C_1 and C_2:

$$C_1 + C_2 = 0, \quad -4C_1 + C_2 = 5,$$

so $C_1 = -1$ and $C_2 = 1$.

The differential equation of order 2

$$\ddot{x} = f(t, x, \dot{x}) \tag{5}$$

can be transformed into a system of two order 1 DE's by introducing new variables:

$$x_1(t) = x(t),$$
$$x_2(t) = \dot{x}(t).$$

Now DE (5) becomes

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = f(t, x_1(t), x_2(t)).$$

An initial condition

$$x(t_0) = \alpha_0, \quad \dot{x}(t_0) = \alpha_1$$

is transformed into an initial condition

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}.$$

In the same way a differential equation of order n

$$x^{(n)} = f(t, x, \dot{x}, \ldots, x^{(n-1)})$$

can be transformed into a system of n differential equations of order 1 by introducing new dependent variables

$$\begin{aligned} x_1 &= x, \\ x_2 &= \dot{x}, \\ &\vdots \\ x_n &= x^{(n-1)}, \end{aligned} \tag{6}$$

and hence (6) becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(t, x_1, x_2, \ldots, x_n) \end{bmatrix}.$$

Example: transformating DEs of higher order into a system of 1st order equations

We are given the differential equation of order 2

$$2\ddot{x} - 5\dot{x} + x = 0, \tag{7}$$

with initial conditions

$$x(3) = 6, \qquad \dot{x}(3) = -1. \tag{8}$$

We introduce new variables:

$$\begin{aligned} x_1(t) &= x(t), \\ x_2(t) &= \dot{x}(t), \end{aligned}$$

and hence (7) becomes the system

$$\begin{aligned} \dot{x_1}(t) &= x_2(t), \\ \dot{x_2}(t) &= \frac{5}{2}x_2 - \frac{1}{2}x_1. \end{aligned}$$

An initial conditions (8) becomes

$$x_1(3) = 6, \qquad x_2(3) = -1.$$

Numerical methods for a system of DEs work exactly in the same way as for a single equation, with the exception that the unknown function is a vector function

$$x(t) = \begin{bmatrix} x_1(t) & \cdots & x_n(t) \end{bmatrix}^T .$$

Given the system with initial condition

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{bmatrix}, \quad x(t_0) = x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix},$$

we construct a recursive sequence of points

$$t_i = t_0 + ih, \quad x^{(i)} \doteq x(t_i), i \geq 0$$

where the vector $x^{(i)}$ is an approximation to the value of the exact solution $x(t_i)$, and h is the step size.

Euler's method:

$$t_{i+1} = t_i + h, \quad x^{(i+1)} = x^{(i)} + hf(t_i, x^{(i)}), \quad i \geq 0.$$

Runge Kutta:

$$t_{i+1} = t_i + h, \quad x^{(i+1)} = x^{(i)} + (k_1 + 2k_2 + 2k_3 + k_4)/6,$$

where

$$\begin{aligned} k_1 &= hf(t_i, x^{(i)}), \\ k_2 &= hf(t_i + h/2, x^{(i)} + k_1/2), \\ k_3 &= hf(t_i + h/2, x^{(i)} + k_2/2), \\ k_4 &= hf(t_i + h, x^{(i)} + k_3). \end{aligned}$$

A system of DEs is *autonomous* if the function $f : \mathbb{R}^n \to \mathbb{R}^n$ does not depend on t:

$$\dot{x} = f(x).$$

For an autonomous system, the tangent vector to a solution depends only on the point x and is independent of the time t at which the solution reaches a given point. In this case, the tangent vectors can be viewed as a *directional field* in the space $\mathbb{R}^n$.

In case of an autonomous system of 2 DE's:

$$f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\dot{x} = f_1(x, y),$$
$$\dot{y} = f_2(x, y),$$

gives a directional field in the (x, y) plane, which we call the *phase plane* of the system.

The general solution is a family of parametric curves or *trajectories* which respect the given directional field at every point (x, y).

The points where $f(x) = 0$ are *stationary points* or *equlibrium points* of the system.

At a stationary point $x_0 = x(t_0)$, $\dot{x}(t_0) = 0$, so $x(t) = x_0$ represents a constant, or equilibrium solution of the system.

The *predator-prey* or *Volterra-Lotka* model is a famous system of DE's proposed by Alfred J. Lotka (1920) for modelling certain chemical reactions, and independently by Vito Volterra (1926) for dynamics of biological systems. It was later applied in economics and is used in a number of domains.

Two populations of species, for example rabbits and foxes, live together and depend on each other.

The number of rabbits (the prey) at time t is $R(t)$ and the number of foxes (the predators) is $F(t)$.

If they live apart, the rabbit, resp. fox, population grows, resp. declines, with the exponential law:

$$\dot{R} = kR, \quad k > 0, \quad \text{resp.} \quad \dot{F} = -rF, \quad r > 0.$$

If they live together, then interactions between rabbits and foxes cause a decline in the rabbit population and a growth of the fox population.
This (basic) predator-prey model is the following:

$$\dot{R} = kR - aRF, \quad \dot{F} = -rF + bFR, \quad a, b > 0.$$

The system has two stationary or equilibrium points:

$$kR - aRF = -rF + bFR = 0 \Rightarrow$$

$$\Rightarrow R = F = 0 \qquad \text{or} \qquad R = \frac{r}{b}, F = \frac{k}{a}.$$

The meaning of these values is that the populations (ideally) coexist peacefully, with no fluctuations in the population sizes.

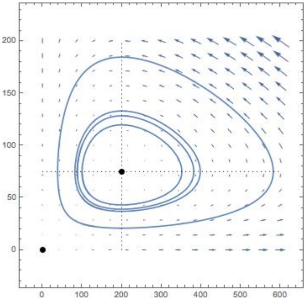
Example of autonomous system

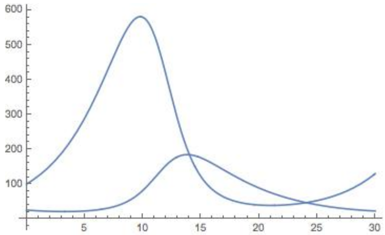
The left figure below shows the directional field and several solutions for the system

$$\dot{R} = 0.3R - 0.004RF \quad \dot{F} = -0.2F + 0.001FR$$

in the (R, F) plane.

The right figure shows dynamics of the population sizes $F(t)$ and $R(t)$ with respect to t:





On the figure below, the blue curve is the exact solution and the black dots are approximations for function values for the predator prey system

$$\dot{R} = 0.3R - 0.004RF \quad \dot{F} = -0.2F + 0.001FR$$

with initial condition

$$R(0) = 500, F(0) = 50$$

using Euler's method with step size $h = 0.5$: