Mathematical modelling

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4.2.4 The dynamics of systems of 2 equations

For an autonomous linear system

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2, \quad \dot{x}_2 = a_{21}x_1 + a_{22}x_2,$$

the origin $(0, 0)$ is always a stationary point, i.e., an equilibrium solution.

The eigenvalues of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

determine the type of the stationary point $(0, 0)$ and the shape of the phase portrait.

We will assume that $\det A \neq 0$. Let λ_1, λ_2 be the eigenvalues of A. We also assume that there exist two linearly independent vectors v_1, v_2 of A (even if $\lambda_1 = \lambda_2$).

Case 1: $\lambda_1, \lambda_2 \in \mathbb{R}$

The general solution is

$$x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2.$$

- If $C_1 = 0$, the trajectory $x_1(t)$ is a ray in the direction of v_2 if $C_2 > 0$, or $-v_1$ if $C_2 < 0$.
- Similarly, if $C_2 = 0$ the trajectory $x_2(t)$ is a ray in the direction of v_2 or $-v_2$.
- The behaviour of other trajectories depends on the signs of λ_1 and λ_2.

Subcase 1.1: $0 < \lambda_1 < \lambda_2$

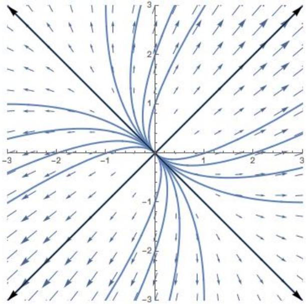
- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_1$.

The point $(0, 0)$ is a *source*.

Example

The general solution of the system $\dot{x}_1 = 3x_1 + x_2, \dot{x}_2 = x_1 + 3x_2$ is

$$x(t) = C_1 e^{4t} \begin{bmatrix} 1 & 1 \end{bmatrix}^T + C_2 e^{2t} \begin{bmatrix} -1 & 1 \end{bmatrix}^T.$$



Subcase 1.2: $\lambda_2 < \lambda_1 < 0$

- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_1$.

The point $(0, 0)$ is a *sink*.

Subcase 1.3: $\lambda_1 < 0 < \lambda_2$

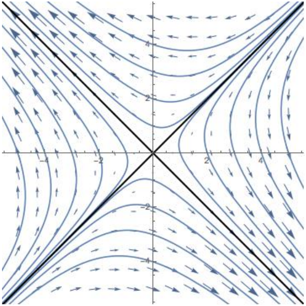
- as $t \to \infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_2 t} v_2$,
- as $t \to -\infty$, $x(t)$ asymptotically approaches the solution $\pm e^{\lambda_1 t} v_1$.

The point $(0, 0)$ is a *saddle*.

Example

The general solution of the system $\dot{x}_1 = x_1 - 3x_2, \dot{x}_2 = -3x_1 + x_2$ is

$$x(t) = C_1 e^{-2t} \begin{bmatrix} 1 & 1 \end{bmatrix}^T + C_2 e^{4t} \begin{bmatrix} 1 & -1 \end{bmatrix}^T.$$



Subcase 2.1: $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha \neq 0$

The general solution is

$$x(t) = e^{\alpha t}\left[(C_1\cos(\beta t) + C_2\sin(\beta t))u + (-C_1\sin(\beta t) + C_2\cos(\beta t))w\right].$$

Hence,

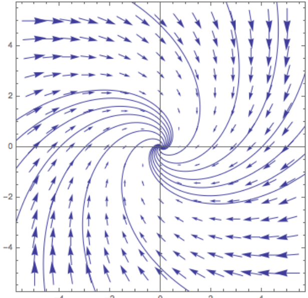
- if $\alpha < 0$, $x(t)$ spirals towards $(0,0)$ as $t \to \infty$, and
- if $\alpha > 0$, $x(t)$ spirals away from $(0,0)$ as $t \to \infty$.

The point $(0,0)$ is a *spiral sink* in the first case and a *spiral source* in the second case.

Example

$$\dot{x}_1 = -3x_1 + 2x_2, \dot{x}_2 = -x_1 - x_2$$

$$x(t) = e^{-2t}\cdot \left((C_1\cos t + C_2\sin t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-C_1\sin t + C_2\cos t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$



Subcase 2.2: $\lambda_{1,2} = \pm i\beta$, $\alpha \neq 0$

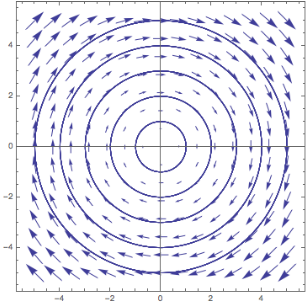
The trajectories are periodic with period $2\pi/\beta$, i.e. the point $x(t)$ circles around $(0,0)$.

The point $(0,0)$ is a *center*.

Example

$$\dot{x} = v, \quad \dot{v} = -\omega^2 x$$

$$x(t) = \\ (C_1 \cos(\omega t) + C_2 \sin(\omega t)) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \\ (-C_1 \sin(\omega t) + C_2 \cos(\omega t)) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Linear DE's of order n

A linear DE (LDE) of degree n is of the form

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = f(t)$$

The equation is

- *homogeneous* if $f(t) = 0$, and
- *nonhomogeneous* if $f(t) \neq 0$.
- The general solution of the homogeneous part is the family of all linear combinations
$$y(t) = C_1 x_1(t) + \cdots + C_n x_n(t)$$
of n linearly independent solutions $x_1(t), \ldots, x_n(t)$.
- If the coefficients $a_1(t), \ldots, a_n(t)$ are continuous functions, then for any $\alpha_0, \ldots, \alpha_n$ there exists exactly one solution satisfying the initial condition
$$x(t_0) = \alpha_0, \quad \dot{x}(t_0) = \alpha_1, \quad \ldots, \quad x^{(n-1)}(t_0) = \alpha_n.$$

Assume that the coefficient functions $a_1(t), \dots a_n(t)$ in a homogeneous LDE are constant:

$$x^{(n)} + a_{n-1}x^{(n-1)} \cdots + a_0 x = 0, \quad a_1, \dots a_n \in \mathbb{R}$$

Then a solution is of the form $x(t) = e^{\lambda t}$ where λ is a root of the *characteristic polynomial*:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0.$$

A polynomial of degree n has exactly n roots, counted by multiplicity. These can be either real or complex conjugate pairs.

From the roots of the characteristic polynomial, n linearly independent solutions of the LDE can be reconstructed.