Mathematical modelling

Lecture 13, May 14, 2021

Faculty of Computer and Information Science
University of Ljubljana

2020/2021

Linear DE's of order n

A linear DE (LDE) of degree n is of the form

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = f(t). \tag{1}$$

The equation is

- *homogeneous* if $f(t) = 0$, and
- *nonhomogeneous* if $f(t) \neq 0$.
- The general solution of the homogeneous part is the family of all linear combinations
$$y(t) = C_1x_1(t) + \cdots + C_nx_n(t)$$
of n linearly independent solutions $x_1(t), \ldots, x_n(t)$.
- If the coefficients $a_1(t), \ldots, a_n(t)$ are continuous functions, then for any $\alpha_0, \ldots, \alpha_n$ there exists exactly one solution satisfying the initial condition
$$x(t_0) = \alpha_0, \quad \dot{x}(t_0) = \alpha_1, \quad \ldots, \quad x^{(n-1)}(t_0) = \alpha_n.$$

Test for linear independence of solutions

Let $x_1(t), \dots, x_n(t)$ be the solutions of the homogeneous part of (1) and form a matrix

$$W(x_1(t), \dots, x_n(t)) := \begin{bmatrix} x_1(t) & \dots & x_n(t) \\ \dot{x}_1(t) & \dots & \dot{x}_n(t) \\ \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & \dots & x_n^{(n-1)}(t) \end{bmatrix}$$

We call the determinant

$$\phi(t) = \det W(x_1(t), \dots, x_n(t)) : I \to \mathbb{R}$$

the *Wronskian determinant* of $W(x_1(t), \dots, x_n(t))$, where I is the interval on which t lives.

Let π_n be the set of all permutations of the set $\{1, \ldots, n\}$. Now we differentiate $\phi(t)$ and obtain

$$\begin{aligned}
\phi'(t) &= \left(\sum_{\sigma \in \pi_n} x_{\sigma(1)} x^{(1)}_{\sigma(2)} \cdots x^{(n-1)}_{\sigma(n)} \right)' (t) \\
&= \sum_{\sigma \in \pi_n} \Big(\left(x_{\sigma(1)} \right)' (t) x^{(1)}_{\sigma(2)}(t) \cdots x^{(n-1)}_{\sigma(n)}(t) \\
&\qquad + x_{\sigma(1)}(t) \left(x^{(1)}_{\sigma(2)} \right)' (t) \cdots x^{(n-1)}_{\sigma(n)}(t) + \cdots \\
&\qquad + x_{\sigma(1)}(t) x^{(1)}_{\sigma(2)}(t) \cdots \left(x^{(n-1)}_{\sigma(n)}(t) \right)' \Big) \\
&= \left(\sum_{\sigma \in \pi_n} x^{(1)}_{\sigma(1)}(t) x^{(1)}_{\sigma(2)}(t) \cdots x^{(n-1)}_{\sigma(n)}(t) \right) + \\
&\qquad \left(\sum_{\sigma \in \pi_n} x_{\sigma(1)}(t) x^{(2)}_{\sigma(2)}(t) x^{(2)}_{\sigma(3)}(t) \cdots x^{(n-1)}_{\sigma(n)}(t) \right) + \\
&\qquad \cdots + \left(\sum_{\sigma \in \pi_n} x_{\sigma(1)}(t) x^{(1)}_{\sigma(2)}(t) \cdots x^{(n-2)}_{\sigma(n)}(t) x^{(n)}_{\sigma(n)}(t) \right).
\end{aligned}$$

Now notice that the first $n-1$ summand are the determinants of the matrices

$$\begin{bmatrix} x_1(t) & \dots & x_n(t) \\ \vdots & \ddots & \vdots \\ x_1^{(i)}(t) & \dots & x_n^{(i)}(t) \\ x_1^{(i)}(t) & \dots & x_n^{(i)}(t) \\ \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & \dots & x_n^{(n-1)}(t) \end{bmatrix} \tag{2}$$

and hence are equal to 0.

For the last summand use the initial DE (1) to express

$$x_{\sigma(n)}^{(n)} := -a_{n-1}(t)x_{\sigma(n)}^{(n-1)} - \dots - a_0(t)x_{\sigma(n)}.$$

The summands of the from $-a_i(t)x_{\sigma(n)}^{(i)}$ for $i < n-1$ give 0 terms in the sum $\sum_{\sigma\in\pi_n}$ since the sum is just the $-a_i(t)$ multiple of the determinant of the form (2), while the term $-a_{n-1}(t)x_{\sigma(n)}^{(n-1)}$ gives

$$-a_{n-1}(t)\phi(t).$$

It follows that $\phi(t)$ satisfies the DE

$$\phi'(t) = -a_{n-1}(t)\phi(t).$$

The solution of this DE is

$$\phi(t) = ke^{-\int a_{n-1}(t)dt}, \quad \text{where} \quad k \in \mathbb{R}.$$

This establishes the following theorem.

Theorem (Existence and uniqueness of solutions)

If $x_1(t), \ldots, x_n(t)$ are solutions of a LDE with continuous coefficient functions $a_1(t), \ldots a_n(t)$, then their Wronskian is either identically equal to 0 or nonzero at every point. In other words, if $W(x_1, \ldots, x_n)$ has a zero at some point t_0, then it is identically equal to 0.

LDEs with constant coefficients

Assume that the coefficient functions $a_1(t), \ldots a_n(t)$ in a homogeneous LDE are constant:

$$x^{(n)} + a_{n-1}x^{(n-1)} \cdots + a_0 x = 0, \quad a_1, \ldots a_n \in \mathbb{R} \tag{3}$$

Translating (3) to the system by the usual trick of introducing new variables

$$x_1 = x, \quad x_2 = x_1', \quad x_3 = x_2', \quad \cdots, \quad x_n = x_{n-1}',$$

(3) becomes

$$x_n' = -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n,$$

or matricially $\vec{x}' = A\vec{x}$:

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \vec{x}(t)$$

- The solutions to this system are of the form
$$x(t) = p_k(t)e^{\lambda t}v,$$
where λ is the eigenvalue of A, $p_k(t)$ is a polynomial of degree k in t and v is the generalized eigenvector. (This follows most easily by the use of the Jordan form of the matrix.)
- In particular, if there are n linearly independent eigenvectors of the matrix A, then all polynomials p_k are constants and generalized eigenvectors are usual eigenvectors.
- By a simple calculation of expressing the determinant of $A - \lambda I$ according to the coefficients and cofactors of the last row, it turns out that the eigenvalues of A are precisely the roots of the *characteristic polynomial* corresponding to (3):
$$P(\lambda) := \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0. \tag{4}$$
- A (trivial) fact with a nontrivial proof, called the *fundamental theorem of algebra*, states that a polynomial of degree n has exactly n roots, counted by multiplicity. In case the matrix A is real, these roots are real or complex conjugate pairs.

- From the roots of the characteristic polynomial (4), n linearly independent solutions of the LDE can be reconstructed.
- For every real root $\lambda \in \mathbb{R}$,

$$x(t) = e^{\lambda t}$$

  is a solution of the homogeneous LDE.
- For a complex conjugate pair of roots $\lambda = \alpha \pm i\beta$, the real and imaginary parts of the complex-valued exponential functions

$$e^{(\alpha \pm i\beta)t} = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$$

  are two linearly independent solutions

$$x_1 = e^{\alpha t} \cos(\beta t), \quad x_2 = e^{\alpha t} \sin(\beta t).$$

- If a root (or a complex pair of roots) λ has multiplicity $k > 1$, then it can be shown that

$$e^{\lambda t}, te^{\lambda t}, \ldots t^{k-1} e^{\lambda t}$$

  are all linearly independent solutions.

Let us prove the last fact by an interesting trick. We introduce the operator

$$L : \mathcal{C}^{(n)}(I) \to \mathcal{C}(I)$$
$$L(u) = u^{(n)} + a_{n-1}u^{(n-1)} \cdots + a_0 u,$$

where $\mathcal{C}^{(n)}(I)$ stands for the vector space of n-times continuously differentiable functions on the interval I and $\mathcal{C}(I)$ stands for the vector space of continuous functions on I.

Let λ_0 be the root of the characteristic polynomial (4) of multiplicity k. Let $0 \leq q \leq k$ by an integer. We will check that $t^q e^{\lambda t}$ solves (3). Notice that

$$t^q e^{\lambda t} = \frac{d^q}{d\lambda^q} e^{\lambda t}.$$

Now

$$L(t^q e^{\lambda t}) = \sum_{i=0}^{n} a_i \left(\frac{d^q}{d\lambda^q} e^{\lambda t} \right)^{(i)} = \sum_{i=0}^{n} a_i \frac{d^i}{dt^i} \left(\frac{d^q}{d\lambda^q} e^{\lambda t} \right)$$
$$= \frac{d^q}{d\lambda^q} \left(\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} e^{\lambda t} \right) = \frac{d^q}{d\lambda^q} \left(\sum_{i=0}^{n} a_i \lambda^i e^{\lambda t} \right) = \frac{d^q}{d\lambda^q} (P(\lambda) e^{\lambda t}).$$

Since

$$\frac{d^q}{d\lambda^q}(P(\lambda)e^{\lambda t}) = \sum_{i=1}^{q} \frac{d^i}{d\lambda^i}(P(\lambda)) \cdot Q_i(t,\lambda),$$

where $Q_i(t,\lambda)$ are functions of t, λ and

$$\frac{d^i}{d\lambda^i}(P(\lambda_0)) = 0, \quad \text{for} \quad i = 0, \dots, q,$$

it follows that

$$L(t^q e^{\lambda_0 t}) = 0.$$

Second order homogeneous LDE with constant coefficients

We are given a DE

$$a\ddot{x} + b\dot{x} + cx = 0,$$

where $a, b, c \in \mathbb{R}$ are real numbers. We know from the theory above that the general solution is

$$x(t, C_1, C_2) = C_1 x_1(t) + C_2 x_2(t),$$

where $C_1, C_2 \in \mathbb{R}$ are parameters and

1. $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ if the characteristic polynomial has two distinct real roots,
2. $x_1(t) = e^{\alpha t} \cos \beta t$ and $x_2(t) = e^{\alpha t} \sin \beta t$ if the characteristic polynomial has a complex pair $\lambda_{12} = \alpha \pm i\beta$ of roots, and
3. $x_1(t) = e^{\lambda t}$, $x_2(t) = te^{\lambda t}$ if the characteristic polynomial has one double real root.

Nonhomogeneous LDEs

We are given the nonhomogeneous LDE

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = f(t),$$

where $f : I \to \mathbb{R}$ is a nonzero function on the interval I. The following holds:

- If x_1 and x_2 are solutions of the nonhomogeneous equation, the difference $x_1 - x_2$ is a solution of the corresponding homogeneous equation.
- The general solution is a sum

  $$x(t, C_1, C_2) = x_p + x_h = x_p + C_1x_1 + \cdots + C_nx_n,$$

  where x_p is a particular solution of the nonhomogeneous equation and $x_1, \ldots, x_n$ are linearly independent solutions of the homogeneous equation.
- The particular solution can be obtained using the method of "intelligent guessing" or the method of *variation of constants*.

The method of "intelligent guessing" typically works if the function $f(t)$ belongs to a class of functions that is closed under derivations, like polynomials, exponential functions and sums of these.

Example ($\ddot{x} + \dot{x} + x = t^2$)

We are guessing that the particular solution will be of the form

$$x_p(t) = At^2 + Bt + C.$$

We have that

$$\dot{x}_p(t) = 2At + B, \quad \ddot{x}_p(t) = 2A,$$

and so

$$\begin{aligned} \ddot{x} + \dot{x} + x &= 2A + (2At + B) + (At^2 + Bt + C) \\ &= At^2 + (2A + B)t + (2A + B + C) \end{aligned}$$

The initial DE gives us a linear system in A, B, C :

$$A = 1, \quad 2A + B = 0, \quad 2A + B + C = 0$$

with the solution $A = 1, B = -2, C = 0$. Hence, $x_p(t) = t^2 - 2t$.

Example ($\ddot{x} - 3\dot{x} + 2x = e^{3t}$)

We are guessing that the particular solution will be of the form

$$x_p(t) = Ae^{3t}.$$

We have that

$$\dot{x}_p(t) = 3Ae^{3t}, \quad \ddot{x}_p(t) = 9Ae^{3t},$$

and so

$$\ddot{x} - 3\dot{x} + 2x = 9Ae^{3t} - 3(3Ae^{3t}) + 2Ae^{3t} = 2Ae^{3t}$$

The initial DE gives us an equation $2A = 1$ and hence, $x_p(t) = \frac{1}{2}e^{3t}$.

Example ($\ddot{x} - x = e^t$)

The particular solution will not be of the form $x_p(t) = Ae^t$, since this is a solution of the homogeneous equation, we are guessing that the correct form in this case is

$$x_p(t) = Ate^t.$$

We have that

$$\dot{x}_p(t) = A(e^t + te^t), \quad \ddot{x}_p(t) = A(2e^t + te^t),$$

and so

$$\ddot{x} - x = A(2e^t + te^t) - Ate^t = 2Ae^t.$$

The initial DE gives us an equation $2A = 1$ and hence, $x_p(t) = \frac{1}{2}te^t$.