Mathematical modelling

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Example ($\ddot{x} + x = \frac{1}{\cos t}$)

Let us first solve the homogeneous part $\ddot{x} + x = 0$. The characteristic polynomial is $p(\lambda) = \lambda^2 + 1$ with zeroes

$$\lambda_{1,2} = \pm i = \cos t \pm i \sin t.$$

Hence, real solutions of the DE are

$$x_1(t) = \cos t \qquad \text{and} \qquad x_2(t) = \sin t. \tag{1}$$

So the general solution to the homogeneous part is

$$x(t) = C_1 x_1(t) + C_2 x_2(t), \qquad \text{where} \quad C_1, C_2 \in \mathbb{R} \quad \text{are constants.}$$

Now we are searching for the particular solution $x_p(t)$ of the form

$$x_p(t) = C_1(t) x_1(t) + C_2(t) x_2(t).$$

Thus,

$$\dot{x}_p(t) = \dot{C}_1(t)x_1(t) + C_1(t)\dot{x}_1(t) + \dot{C}_2(t)x_2(t) + C_2(t)\dot{x}_2(t). \quad (2)$$

We force an equation

$$\dot{C}_1(t)x_1(t) + \dot{C}_2(t)x_2(t) = 0. \quad (3)$$

Differentiang (2) further under the assumption (3) we get

$$\ddot{x}_p(t) = (\dot{C}_1(t)\dot{x}_1(t) + C_1(t)\ddot{x}_1(t)) + (\dot{C}_2(t)\dot{x}_2(t) + C_2(t)\ddot{x}_2(t)). \quad (4)$$

Plugging this into the DE and using that x_1, x_2 are solutions of $\ddot{x} + x = 0$

$$\dot{C}_1(t)\dot{x}_1(t) + \dot{C}_2(t)\dot{x}_2(t) = \frac{1}{\cos t}. \quad (5)$$

Expressing $\dot{C}_2(t)$ from (3) and plugging into (5) we get

$$\dot{C}_1(t)\dot{x}_1(t) - \frac{\dot{C}_1(t)x_1(t)}{x_2(t)}\dot{x}_2(t) = \dot{C}_1(t)\frac{\dot{x}_1(t)x_2(t) - x_1(t)\dot{x}_2(t)}{x_2(t)} = \frac{1}{\cos t}. \quad (6)$$

Using (1) in (6) we get

$$\dot{C}_1(t) = -\frac{\sin t}{\cos t}. \quad (7)$$

Hence,

$$C_1(t) = -\int \frac{\sin t}{\cos t} dt = -\int \frac{1}{u} du = -\log|u| = -\log|\cos t|,$$

where we used the substitution $u = \cos t$.

Using (7) in (3) we get

$$\dot{C}_2(t) = 1. \tag{8}$$

Hence,

$$C_2(t) = t.$$

So,

$$x_p(t) = -\log|\cos t| \cdot \cos t + t \sin t.$$

The complete solution to DE is

$$x(t) = C_1 \cos t + C_2 \sin t - \log|\cos t| \cdot \cos t + t \sin t,$$

where C_1, C_2 are parameters.

Vibrating systems

There are many vibrating systems in many different domains. The mathematical model is always the same, though. We will have in mind a vibrating mass attached to a spring.

Case 1: *Free vibrations without damping*

Let $x(t)$ denote the displacement of the mass from the equillibrium position.

- According to **Newton's second law of motion**

$$m\ddot{x} = \sum F_i,$$

  where F_i are forces acting on the mass.

- By **Hooke's law**, the only force acting on the mass pulls towards the equilibrium, its size is proportional to the displacement and the direction is opposite

$$F = -kx(t), \quad k > 0.$$

- So the DE in this case is

$$\boxed{m\ddot{x} + kx = 0}.$$

- The characteristic equation

$$m\lambda^2 + k = 0$$

  has complex solutions $\lambda = \pm\omega i$, $\omega^2 = k/m$.
- The general solution is

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t.$$

- So the solutions $x(t)$ are periodic. The equillibrium point $(0, 0)$ in the phase plane (x, v) is a *center*.

Case 2: *Free vibrations with damping*

We assume a linear damping force

$$F_d = -\beta \dot{x},$$

so the DE is

$$\boxed{m\ddot{x} + \beta\dot{x} + kx = 0}, \quad \text{where} \quad m, \beta, k > 0.$$

Depending on the solutions of the characteristic equation there are three cases:

- *Overdamping* when $D = \beta^2 - 4km > 0$ and $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, $\lambda_{1,2} < 0$. The mass slides towards the equilibrium. The point $(0,0)$ in the (x, v) plane is a *sink*.
- *Critical damping* when $D = 0$ and $x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$, $\lambda < 0$. The point mass slides towards the equillibrium after, possibly, one swing. The point $(0,0)$ in the (x, v) plane is a *sink*,
- *Damped vibration* when $D < 0$ and $x(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$. The mass oscillates around the equillibrium with decreasing amplitudes. The point $(0,0)$ is a *spiral sink*.

Case 3: *Forced vibration without damping*

In addition to internal forces of the system there is an additional external force $f(t)$ acting on the system, so

$$\boxed{m\ddot{x} + kx = f(t)}.$$

The general solution is of the form

$$x(t, C_1, C_2) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + x_p(t),$$

where x_p is a particular solution of the nonhomogeneous equations.

Example

Let $f(t) = a \sin \mu t$.

Using the method of intelligent guessing,

- if $\mu \neq \omega$, then $x_p(t) = A \sin \mu t + B \cos \mu t$
- if $\mu = \omega$, then $x_p = t(A \sin \omega t + B \cos \omega t)$, so the solutions of the equation are unbonded and incerase towards ∞ as $t \to \infty$ – the well known phenomenion of *resonance* occurs.

Case 4: *Forced vibration with damping:*

$$\boxed{m\ddot{x} + \beta\dot{x} + kx = f(t)}.$$

Example

Let $f(t) = a\sin\mu t$.
The general solution is of the form

$$x(t, C_1, C_2) = x_h C_1 x_1(t) + C_2 x_2(t) + x_p(t)$$

where $x_p(t)$ is of the form $A\sin\mu t + B\cos\mu t$, and the two solutions x_1 and x_2 both converge to 0 as $t \to \infty$. For any C_1, C_2 the solution $x(t, C_1, C_2)$ asymptotically tends towards $x_p(t)$.