Please show **all** your work! Answers without supporting work will not be given credit. Be concise and precise! You have 75 minutes to complete this exam. To pass: a minimum of eight (8) ❀ and four (4) ♥ . Grading: ≥13pt = 7, ≥14pt = 8, ≥16pt = 9, otherwise 6. Optional oral test if ≥16pt (+0 or +1 grade).

(1) Let $(\Omega, 2^\Omega)$, $\Omega = \{1, 2, ..., 10\}$. Is $\mu(A) = |A|^2$, $A \in 2^\Omega$, a measure on $(\Omega, 2^\Omega)$? (1pt, ❀)

(2) Write Boole's inequality (also known as the union bound). (1pt, ❀)

(3) Give a concrete example of an algebra that is not a σ-algebra. (1pt, ❀)

(4) Write the def. of an abstract integral of a non-negative simple function. (1pt, ❀)

(5) Write the properties of a cumulative distribution function (CDF). (1pt, ❀)

(6) Write the probability mass function (PMF) of the binomial distribution. (1pt, ❀)

(7) Show that an estimator can be unbiased but not consistent. (1pt, ❀)

(8) Write the definition of correlation. (1pt, ❀)

(9) Write the central limit theorem. (1pt, ❀)

(10) Write Markov's inequality. (1pt, ❀)

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(11) We have n integer observations $y_1, ..., y_n$ and we assume that they come from a Poisson distribution with unknown rate λ: $y_i \sim_{\text{iid}} \text{Poisson}(\lambda)$.

(a) Derive the maximum likelihood estimator θ_{MLE} of θ. (1pt, ♥)

(b) Derive Fisher information. (1pt, ♥)

(12) We have two independent standard normal RVs X and Y. Let $Z = 2X - Y$ and $W = -X + Y$. Find f_{ZW} the joint PDF of RVs Z and W. You may follow these steps for partial credit:

(a) Determine the transformation and find its inverse. (1pt, ♥)

(b) Find the Jacobian of the transformation (or inverse). (1pt, ♥)

(c) Find f_{ZW} using the rule for transforming jointly continuous RVs. (1pt, ♥)

(13) Let X and Y be independent Uniform(0, 1). Find the CDF and PDF of $Z = XY$. (1pt, ♥)

(14) Show that the Normal distr. is a conjugate prior for the Normal likelihood mean parameter for the case where the variance is a known constant (not a parameter!). (1pt, ♥ , ❀)

(15) We have a system with 200 components. In the observed time period a component will fail with probability $\frac{1}{10}$, independent of other components. There is some redundancy in the system - the system will fail if and only if 100 or more of its components fail. Provide an upper bound for the probability of the system failing using Chebyshev's inequality. (1pt, ♥)

(16) If X is uniformly distributed over $(0, 1)$, find the density function of $Y = e^X$. (1pt, ♥)

(17) Let a be drawn from Uniform(0, 1). Compute the expectation $E[\max\{a, 1 - a\}]$. (1pt, ♥)